Chapter seven

Application of integrals

7-1- <u>Definite integrals</u>:

If $f(x)$ is continuous in the interval $a \le x \le b$ and it is integrable in the interval then the area under the curve:-

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:-

1) $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$, *where* c *is constant.*

2) $\int_a^b \left(f(x) \mp g(x)\right)dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) *Let* $a < c < b$ *then* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5) $\int_a^a f(x)\,dx = 0$

6) *If* $f(x) \ge 0$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \ge 0$

7) *If* $f(x) \le g(x)$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

EX-1 **– Evaluate the following definite integrals:**

$$1)\ \int_{2}^{6} \frac{dx}{x+2} \qquad 2)\ \int_{\pi/2}^{3\pi/2} \cos x\,dx$$

$$3)\ \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2} \qquad 4)\ \int_{0}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$$

$$5)\ \int_{-2}^{4} e^{-\frac{x}{2}}\,dx \qquad 6)\ \int_{0}^{\pi} (\pi - x)\cdot \cos x\,dx$$

Sol. –

$$1)\ \int_{2}^{6} \frac{dx}{x+2} = \ln(x+2)\Big|_{2}^{6} = \ln(6+2) - \ln(2+2) = \ln 8 - \ln 4 = 3\ln 2 - 2\ln 2 = \ln 2$$

$$2)\ \int_{\pi/2}^{3\pi/2} \cos x\,dx = \sin x\Big|_{\pi/2}^{3\pi/2} = \sin(\frac{3}{2}\pi) - \sin(\frac{\pi}{2}) = -1 - 1 = -2$$

$$3)\ \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}\Big|_{-\sqrt{3}}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}(-\sqrt{3}) = \frac{\pi}{3} - (-\frac{\pi}{3}) = \frac{2}{3}\pi$$

$$4)\ \int_{0}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_{0}^{\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

$$5)\ \int_{-2}^{4} e^{-\frac{x}{2}}\,dx = -2e^{-\frac{x}{2}}\Big|_{-2}^{4} = -2(e^{-2} - e) = 2(e - e^{-2})$$

$$6)\ \text{Let} \quad u = \pi - x \ \Rightarrow\ du = -dx \quad \& \quad dv = \cos x\,dx \ \Rightarrow\ v = \sin x$$

$$\int_{0}^{\pi} (\pi - x)\cdot \cos x\,dx = (\pi - x)\sin x\Big|_{0}^{\pi} + \int_{0}^{\pi} \sin x\,dx = (\pi - x)\sin x - \cos x\Big|_{0}^{\pi}$$

$$= (\pi - \pi)\sin\pi - \cos\pi - \big((\pi - 0)\sin 0 - \cos 0\big) = 0 - (-1) - (0 - 1) = 2$$

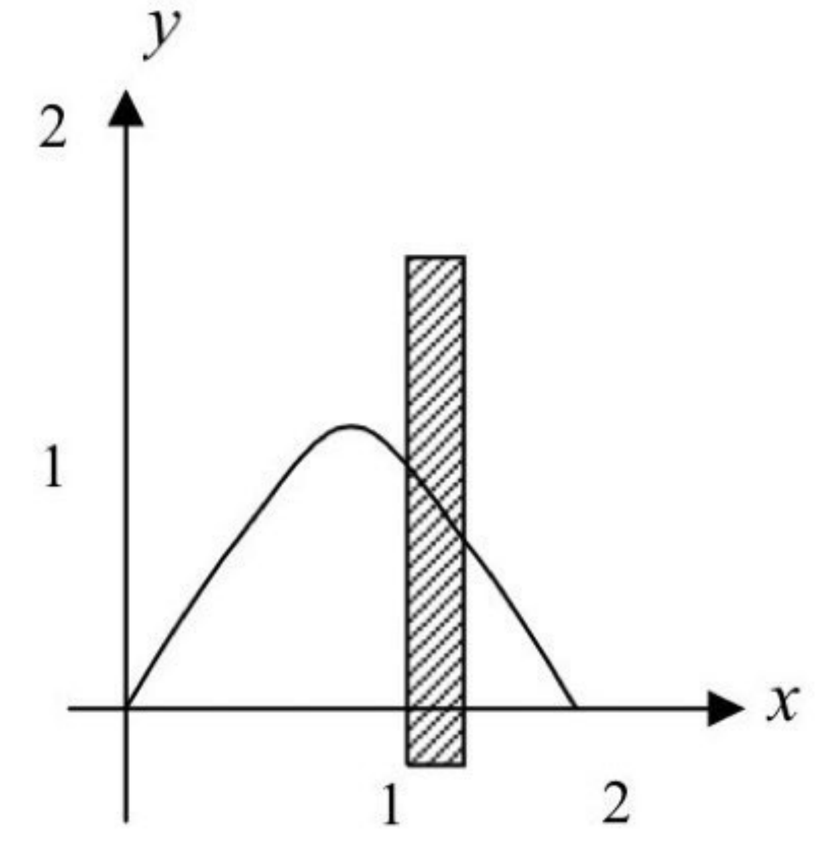
7-2- Area between two curves:

Suppose that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ define two functions of x that are continuous for $a \le x \le b$ then the area bounded above by the y_1 curve, below by y_2 curve and on the sides by the vertical lines $x = a$ and $x = b$ is:-

$$A = \int_a^b [f_1(x) - f_2(x)]\,dx$$

***EX-2*- Find the area bounded by the *x*-axis and the curve:**

$$y = 2x - x^2$$



***Sol.*-**

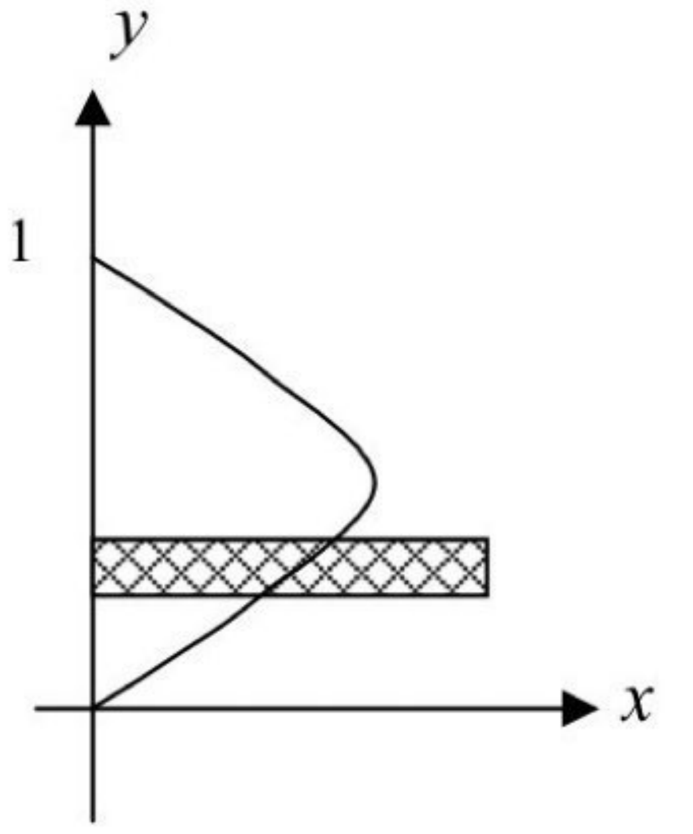
$$\left.\begin{array}{ll} y = 0 & \text{.............(1)} \\ y = 2x - x^2 & \text{.....(2)} \end{array}\right\} \Rightarrow x(x-2) = 0 \Rightarrow x = 0\,,2$$

The points of the intersection of the curve and the *x*-axis are (0,0) and (2,0) then the area bounded by *x*-axis and the curve is:-

$$\int_0^2 (2x - x^2)\,dx = x^2 - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} - (0-0) = \frac{4}{3}$$

***EX-3*- Find the area bounded by the *y*-axis and the curve:**

$$x = y^2 - y^3$$



***Sol.*-**

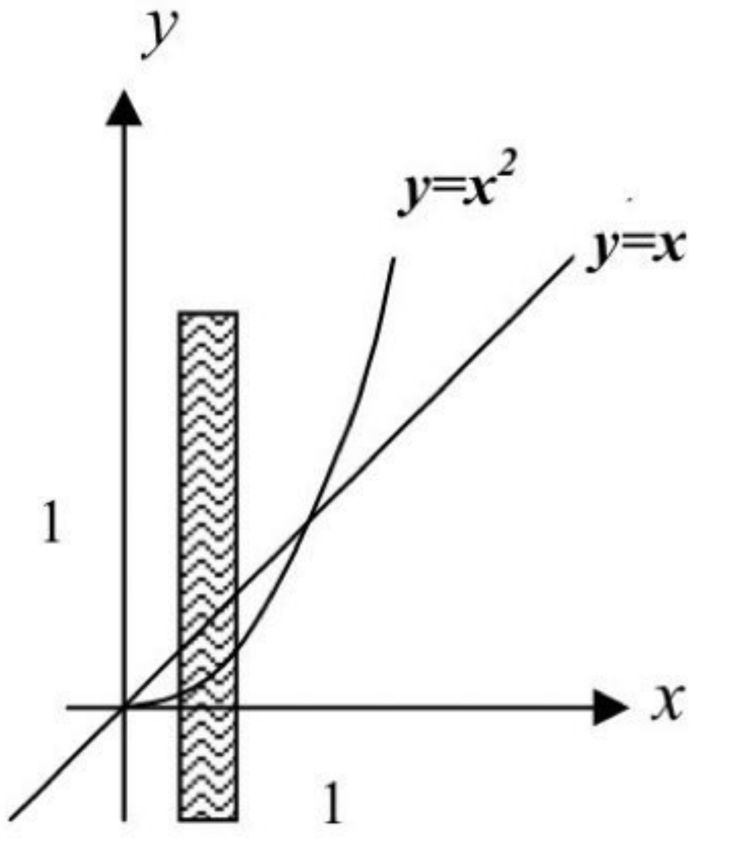
$$\left.\begin{array}{ll} x = 0 & \text{.............(1)} \\ x = y^2 - y^3 & \text{.....(2)} \end{array}\right\} \Rightarrow y^2(1-y) = 0 \Rightarrow y = 0\,,1$$

$$\Rightarrow \textit{intersection points } (0,0), (0,1)$$

The area =

$$A = \int_0^1 (y^2 - y^3)\,dy = \frac{y^3}{3} - \frac{y^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} - (0-0) = \frac{1}{12}$$

EX-4- **Find the area bounded by the curve** $y = x^2$ **and the line:** $y = x$



Sol.-

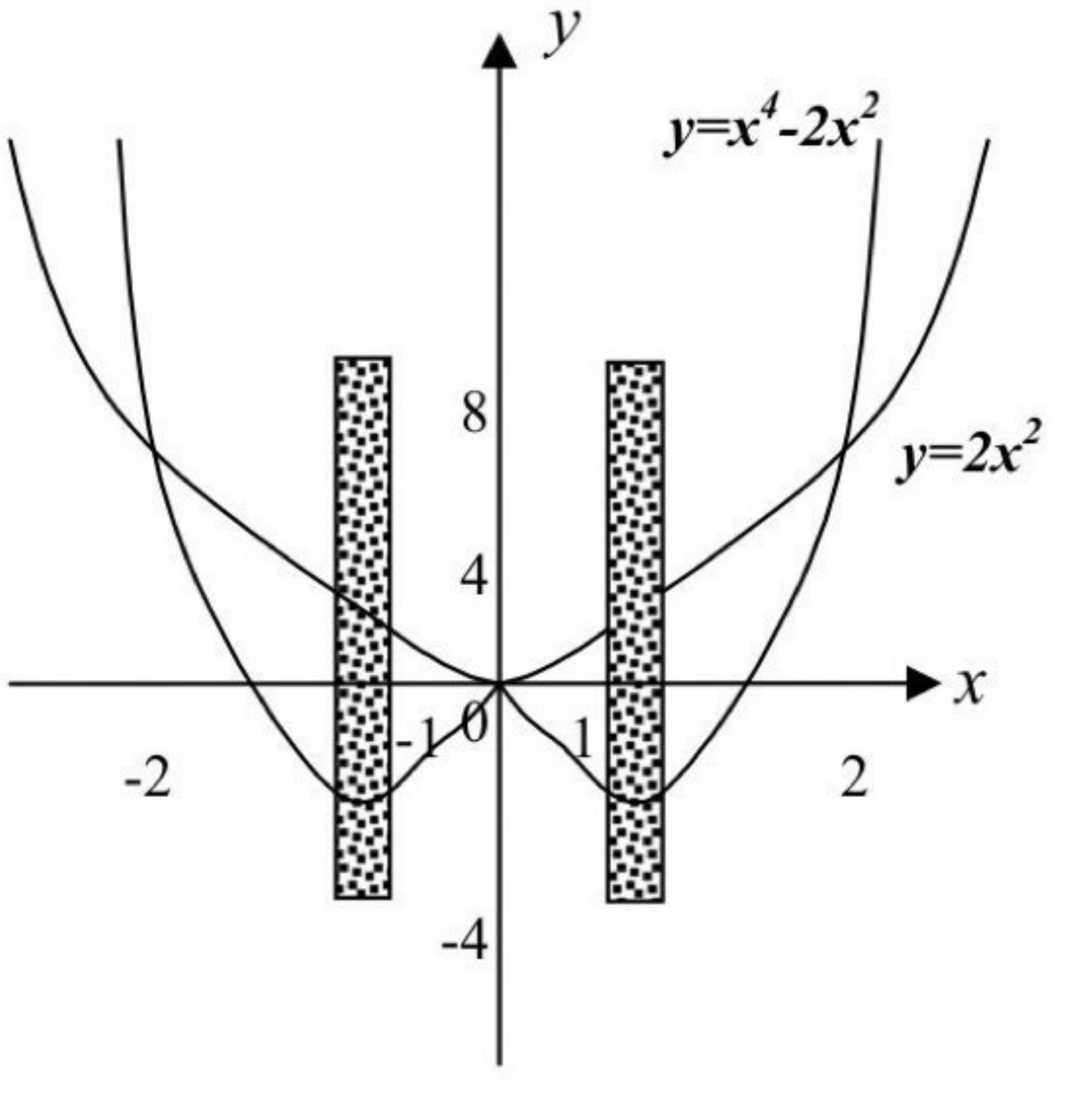
$$\left.\begin{aligned} y &= x^2 \text{ ......(1)} \\ y &= x \text{ .....(2)} \end{aligned}\right\} \Rightarrow x(x-1)=0 \Rightarrow x = 0\,,1$$

$$\Rightarrow \text{ intersection points } (0,0),(1,1)$$

The area =

$$A = \int_0^1 (x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}$$

EX-5- **Find the area bounded by the curves** $y = x^4 - 2x^2$ **and** $y = 2x^2$



Sol.-

$$\left.\begin{aligned} y &= x^4 - 2x^2 \text{ ......(1)} \\ y &= 2x^2 \text{ ....(2)} \end{aligned}\right\} \Rightarrow x^2(x^2-4)=0$$

$$\Rightarrow x = 0\,,2,-2$$

$$\Rightarrow \text{ intersection points are } (0,0),(2,8),(-2,8)$$

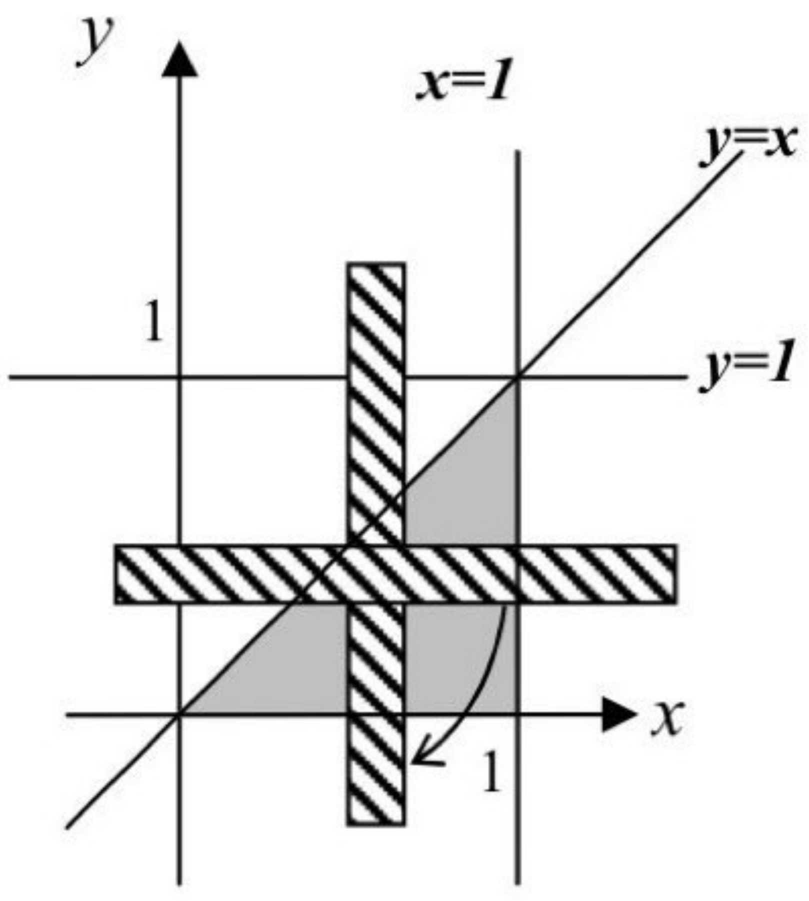
The area =

$$A = \int_{-2}^{0}\left(2x^2 - (x^4 - 2x^2)\right)dx + \int_{0}^{2}\left(2x^2 - (x^4 - 2x^2)\right)dx$$

$$= 2\int_0^2 \left(4x^2 - x^4\right)dx = 2\left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2 = 2\left[\frac{4}{3}\cdot 8 - \frac{32}{5} - 0\right]$$

$$= \frac{128}{15}$$

***EX-7*- Calculate:** $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$



***Sol.*- We cannot solve the integration** $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$**, hence we reverse the order of integration as follow:-**

$x = 1 \quad and \quad y = 1$

$x = y \qquad\quad y = 0$

$$A = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx = \int_0^1 \frac{\sin x}{x}\, y\Big|_0^x\, dx = \int_0^1 \frac{\sin x}{x}(x - 0)\,dx$$

$$= \int_0^1 \sin x\, dx = -\cos x\Big|_0^1 = -(\cos 1 - \cos 0) = 1 - \cos 1$$

***EX-8*- Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.**

$1)\ \int_{-2}^{1} \int_{x^2+4x}^{3x+2} dy\, dx \qquad\qquad 2)\ \int_{-1}^{0} \int_{-2x}^{1-x} dy\, dx + \int_{0}^{2} \int_{-\frac{x}{2}}^{1-x} dy\, dx$

***Sol.*-**

$$1)\quad \left.\begin{aligned} y &= 3x + 2 \ .....(1) \\ y &= x^2 + 4x \ ...(2) \end{aligned}\right\} \Rightarrow$$

$$(x+2)(x-1) = 0$$

$$either\ \ x = -2 \Rightarrow y = -4$$

$$or\quad x = 1 \Rightarrow y = 5$$