<u>Notice:-</u> **We can use the double integration to calculate the area between two curves which bounded above by the curve $y = f_2(x)$ below by $y = f_1(x)$ on the left by the line $x = a$ and on the right by $x = b$, then:-**

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

To evaluate above integrals we follow:-

(a) integrating $\int dy$ with respect to y and evaluating the resulting integral the limits $y = f_1(x)$ and $y = f_2(x)$, then:

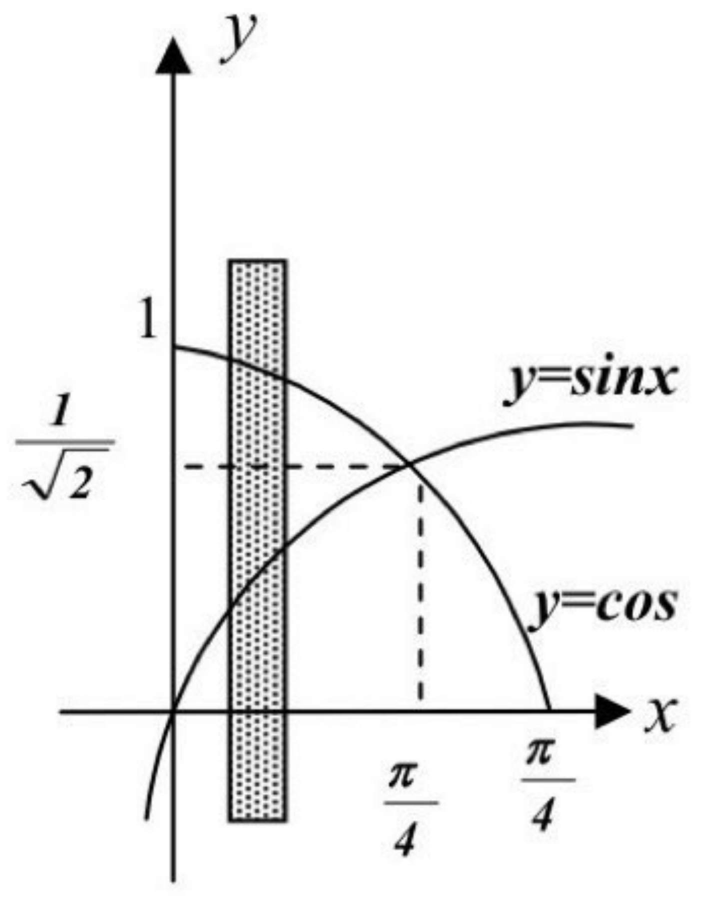
(b) integrating the result of (a) with respect to x between the limits $x = a$ and $x = b$.

If the area is bounded on the left by the curve $x = g_1(y)$, on the right by $x = g_2(y)$, below by the line $y = c$, and above by the line $y = d$, then it is better to integrate first with respect to x and then with respect to y. That is:-

$$A = \int_c^d \int_{g_1(y)}^{g_2(y)} dx\,dy$$

***<u>EX-6</u>*- Find the area of the triangular region in the first quadrant bounded by the *y*-axis and the curve $y = \sin x$, $y = \cos x$.**

***<u>Sol.</u>*-**



$$\left.\begin{aligned} y &= \sin x \;.....(1) \\ y &= \cos x \;.....(2) \end{aligned}\right\} \Rightarrow \sin x = \cos x \qquad \therefore x = \frac{\pi}{4}$$

The area =

$$A = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\,dx = \int_0^{\frac{\pi}{4}} y \Big|_{\sin x}^{\cos x} dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0 + 1) = \sqrt{2} - 1 = 0.414$$

$$(a)\ \int_{-2}^{1}\int_{x^2+4x}^{3x+2} dy\,dx = \int_{-2}^{1} y\Big|_{x^2+4x}^{3x+2} dx = \int_{-2}^{1}(2-x-x^2)dx$$

$$= 2x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-2}^{1} = 2 - \frac{1}{2} - \frac{1}{3} - (-4-2+\frac{8}{3}) = \frac{9}{2}$$

(b) The reversed integral is :-

$$y = 3x+2 \ \Rightarrow\ x = \frac{y-2}{3}$$

$$y = x^2+4x \Rightarrow (x+2)^2 = y+4 \Rightarrow x = -2 \mp \sqrt{y+4}$$

$$Since\ -2 \le x \le 1 \ \Rightarrow\ x = -2+\sqrt{y+4}$$

$$\int_{-4}^{5}\int_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} dx\,dy = \int_{-4}^{5} x\Big|_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} = \int_{-4}^{5}\left(-2+\sqrt{y+4}-\frac{y-2}{3}\right)dy$$

$$= -2y + \frac{2}{3}(y+4)^{3/2} - \frac{(y-2)^2}{6}\Big|_{-4}^{5}$$

$$= -10 + \frac{2}{3}(27) - \frac{9}{6} - (8+0-\frac{36}{6}) = \frac{9}{2}$$

= *The same result as in (a).*

$$2)\ (a)\ \int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-\frac{x}{2}}^{1-x} dy\,dx = \int_{-1}^{0} y\Big|_{-2x}^{1-x} dx + \int_{0}^{2} y\Big|_{-\frac{x}{2}}^{1-x} dx$$

$$= \int_{-1}^{0}(1+x)\,dx + \int_{0}^{2}(1-\frac{x}{2})\,dx = x + \frac{x^2}{2}\Big|_{-1}^{0} + x - \frac{x^2}{4}\Big|_{0}^{2}$$

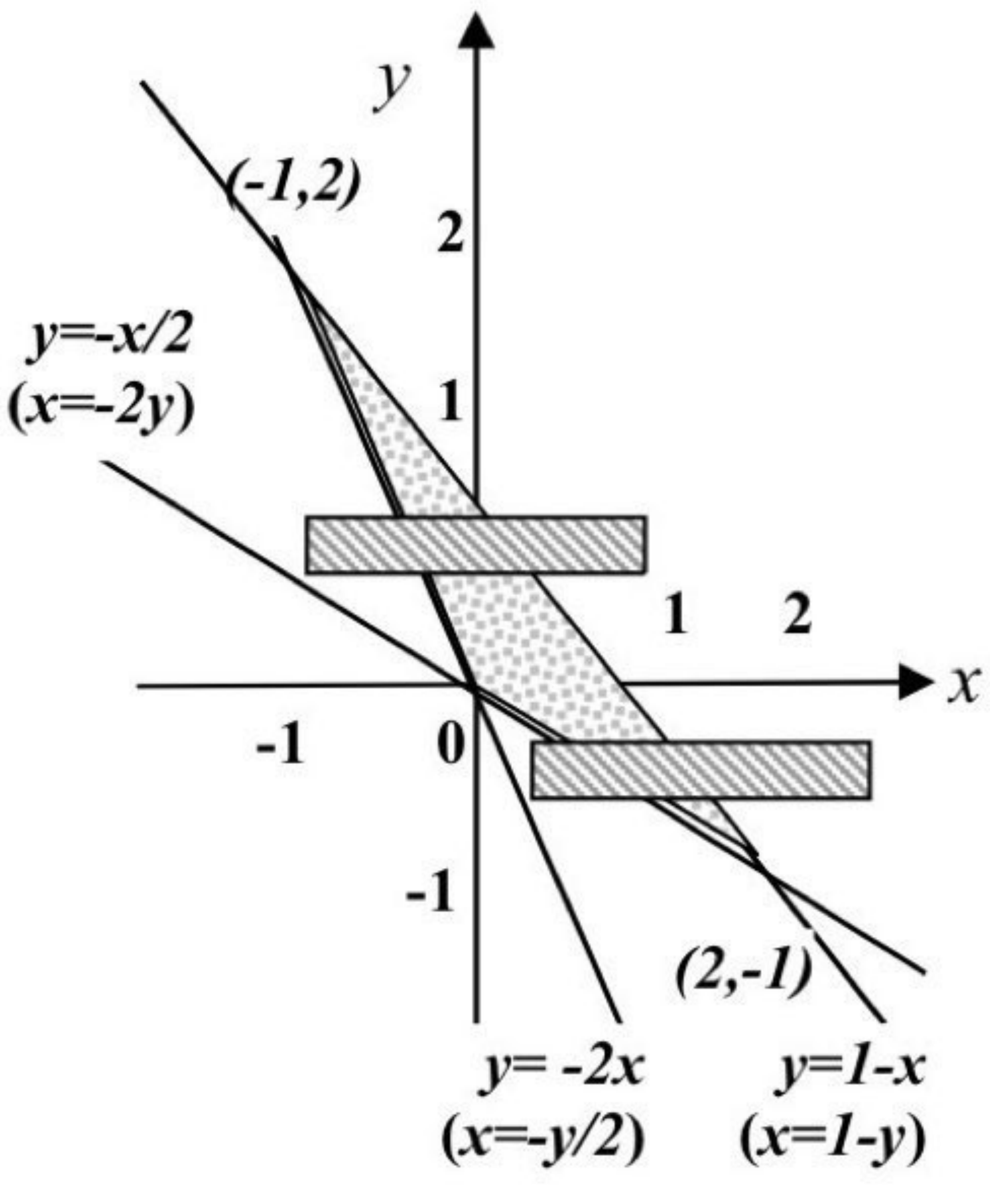
$$= 0 - (-1+\frac{1}{2}) + 2 - 1 - 0 = \frac{3}{2}$$

(b) 1st region

$$\left.\begin{aligned} y &= 1-x\ .....(1) \\ y &= -2x\ ......(2) \end{aligned}\right\} \Rightarrow x = -1 \Rightarrow y = 2 \qquad x\ \text{from}\ -1\ \text{to}\ 0$$

2nd region

$$\left.\begin{aligned} y &= 1-x \quad ...(1) \\ y &= -\frac{x}{2} \quad .....(2) \end{aligned}\right\} \Rightarrow x = 2 \Rightarrow y = -1 \qquad y \text{ from } 0 \text{ to } 2$$



$$\int_0^2 \int_{-\frac{y}{2}}^{1-y} dx\,dy + \int_{-1}^0 \int_{-2y}^{1-y} dx\,dy = \int_0^2 x \Big|_{-\frac{y}{2}}^{1-y} dy + \int_{-1}^0 x \Big|_{-2y}^{1-y} dy$$

$$= \int_0^2 (1-\frac{y}{2})dy + \int_{-1}^0 (1+y)dy = y - \frac{y^2}{4}\Big|_0^2 + y + \frac{y^2}{2}\Big|_{-1}^0$$

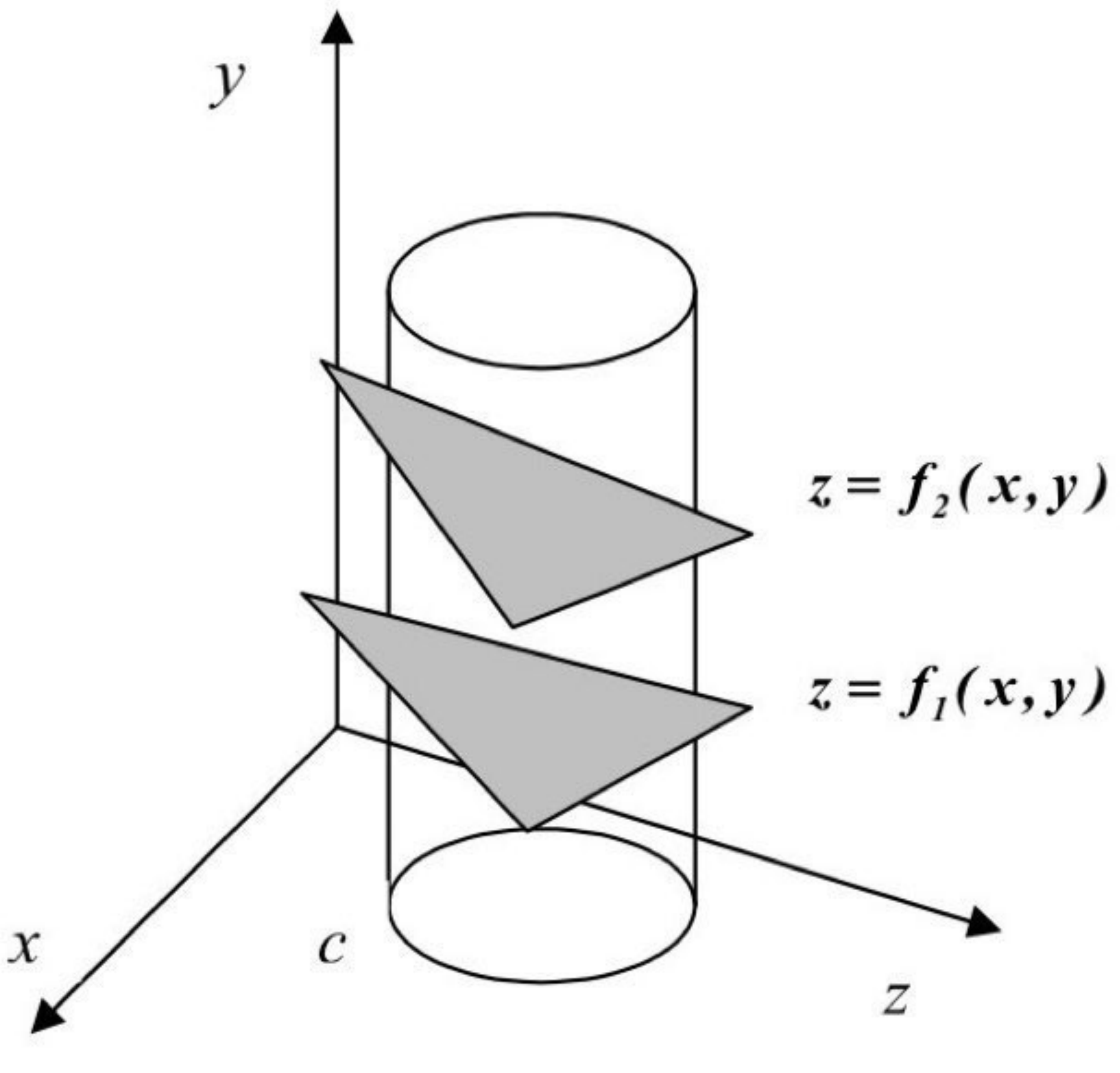
$$= 2-1-0+0-(-1+\frac{1}{2}) = \frac{3}{2}$$

$= \textit{The same result as in (a).}$

7-3- Triple integrals (Volume):

Consider a region N in xyz-space bounded below by a surface $z = f_1(x,y)$, above by the surface $z = f_2(x,y)$ and laterally by a cylinder c with elements parallel to the z-axis. Let A denote the region of the xy-plane enclosed by cylinder c (that is, A is the region covered by the orthogonal projection of the solid into xy-plane). Then the volume V of the region V can be found by evaluating the triply iterated integral:-

$$V = \iint_A \int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx$$



Let z-limits of integration indicate that for every (x,y) in the region A, Z may extend from the lower surface $z = f_1(x,y)$ to the surface $z = f_2(x,y)$. The y- and x-limits of integration have not been given explicitly in equation above, but are indicated as extending over the region A.

We can find the equation of the boundary of the region A by eliminating z between the two equations $z = f_1(x,y)$ and $z = f_2(x,y)$, thus obtaining an equation $f_1(x,y) = f_2(x,y)$ which contains no z, and interpret it as an equation in the xy-plane.

EX-9 **The volume in the first octant bounded by the cylinder $x = 4 - y^2$, and the planes $z = y,\ x = 0,\ z = 0.$**

Sol.-

$$x = 4 - y^2 \Rightarrow y = \mp\sqrt{4-x} \qquad \textit{in first octant :-}$$

$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^y dz\,dy\,dx = \int_0^4 \int_0^{\sqrt{4-x}} z\Big|_0^y dy\,dx = \int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx = \int_0^4 \frac{y^2}{2}\Big|_0^{\sqrt{4-x}} dx$$

$$= \frac{1}{2}\int_0^4 (4 - x - 0)dx = \frac{1}{2}\left[4x - \frac{x^2}{2}\right]_0^4 = \frac{1}{2}\left[16 - \frac{16}{2} - 0\right] = 4$$

EX-10 **The volume enclosed by the cylinders $z = 5 - x^2, z = 4x^2$ and the planes $y = 0,\ x + y = 1.$**

Sol.-

$$\left.\begin{aligned} z &= 5 - x^2 \text{ ...(1)} \\ z &= 4x^2 \text{ ......(2)} \end{aligned}\right\} \Rightarrow x = \mp 1$$

$$V = \int_{-1}^{1} \int_0^{1-x} \int_{4x^2}^{5-x^2} dz\,dy\,dx = \int_{-1}^{1} \int_0^{1-x} z\Big|_{4x^2}^{5-x^2} dy\,dx = \int_{-1}^{1} \int_0^{1-x} (5 - 5x^2)\,dy\,dx$$

$$= 5\int_{-1}^{1} (1 - x^2)\,y\Big|_0^{1-x} dx = 5\int_{-1}^{1} (1 - x^2)(1 - x)dx$$

$$= 5\int_{-1}^{1} (1 - x - x^2 + x^3)\,dx = 5\left[x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\right]_{-1}^{1}$$

$$= 5\left[(1+1) - \frac{1}{2}(1-1) - \frac{1}{3}(1+1) + \frac{1}{4}(1-1)\right] = \frac{20}{3}$$

EX-11 **The volume enclosed by the cylinders** $y^2 + 4z^2 = 16$ **and the planes** $x = 0\ ,\ \ x + y = 4$ **.**

Sol.-

$$y^2 + 4z^2 = 16 \quad \Rightarrow \quad y = \mp 2\sqrt{4 - z^2}$$

$$V = \int_{-2}^{2} \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} \int_{0}^{4-y} dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} (4 - y)\, dy\, dz = \int_{-2}^{2} 4y - \frac{y^2}{2} \Bigg|_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} dz = 16 \int_{-2}^{2} \left(4 - z^2\right)^{1/2} dz$$

$$let \quad z = 2\sin\theta \ \Rightarrow \ dz = 2\cos\theta\, d\theta\ , \quad \theta = \sin^{-1}\frac{z}{2} \quad \overset{at\ z=2\ \Rightarrow\ \theta=\frac{\pi}{2}}{\underset{at\ z=2\ \Rightarrow\ \theta=\frac{\pi}{2}}{\Rightarrow\Rightarrow\Rightarrow}}$$

$$V = 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 - 4\sin^2\theta)^{1/2}\, 2\cos\theta\, d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= 32 \left[\theta + \frac{1}{2}\sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 32 \left[\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{1}{2}(0 - 0) \right] = 32\pi$$

EX-12 **The volume bounded by the ellipse paraboloids** $z = x^2 + 9y^2$ **and** $z = 18 - x^2 - 9y^2$ **.**

Sol.-

$$\left. \begin{aligned} z &= 18 - x^2 - 9y^2 \ \ ..(1) \\ z &= x^2 + 9y^2 \ \ ....(2) \end{aligned} \right\} \Rightarrow \ 9 - x^2 - 9y^2 = 0 \ \Rightarrow \ y = \mp \frac{1}{3}\sqrt{9 - x^2}$$

$$V = \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} \int_{x^2+9y^2}^{18-x^2-9y^2} dz\, dy\, dx = \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} \left[18 - x^2 - 9y^2 - (x^2 + 9y^2) \right] dy\, dx$$