Counting techniques

In the previous lesson, we learned that the classical approach to assigning probability to an event involves determining the number of elements in the event and the sample space. There are many situations in which it would be too difficult and/or too tedious to list all of the possible outcomes in a sample space. when the sample space or event space are very large, that it isn't feasible to write it out. In that case, it helps to have mathematical tools for counting the size of the sample space and event space. These tools are known as counting techniques.

1-The multiplication

If an experiment takes place in k stages with n_i possible outcomes at stage i, the total number of possible outcomes is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$

Example 1: Suppose a password had to be chosen according to strict rules: a lower-case letter first, a digit second, an upper-case letter third. How many ways are there to do this?

solution:

There are 26 lower-case letters, 10 digits, and 26 upper-case letters so the number of such passwords would be 26*10*26 = 6760.

Example 2: A restaurant offers 3 choices for appetizer, 2 for main course, and 5 for dessert. How many meal variations are possible if a diner chooses 1 of each?

solution:

There are 3 × 2 × 5 = 30 different meal variations.

Example 3: How many ways are there to create a 4 digit PIN?

solution:

This is an experiment in 4 stages. At each stage there are 10 possible outcomes (the digits 0, 1, ..., 9) thus there are

$$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$$ possible outcomes for the entire experiment.

Example 4:

Some license plates in Arizona consist of three digits followed by three letters. How many license plates of this type are possible if:

1. There are 10 digits (0, 1, 2, 3, ..., 9) and 26 letters.

$$\underbrace{(10 \cdot 10 \cdot 10) \cdot (26 \cdot 26 \cdot 26)}_{\text{digits}} = 17,576,000 \text{license plates}$$

2. letters can be repeated but digits cannot?

$$\underbrace{(\underline{10} \cdot \underline{9} \cdot \underline{8}) \cdot (26 \cdot 26 \cdot 26)}_{\text{letters}} = 12,654,720 \text{ license plates}$$

3. the first digit cannot be zero and both digits and letters can be repeated?

$$\underbrace{(\underline{9} \cdot \underline{10} \cdot \underline{10})}_{\text{digits}} \cdot \underbrace{(\underline{26} \cdot 26 \cdot 26)}_{\text{letters}} = 15,818,400 \text{ license plates}$$

4. neither digits nor numbers can be repeated.

$$\underbrace{(\underline{10} \cdot \underline{9} \cdot \underline{8})}_{\text{digits}} \cdot \underbrace{(\underline{26} \cdot \underline{25} \cdot \underline{24})}_{\text{letters}} = 11,232,000 \text{ license plates}$$

2- Permutations

The word ‘permutations’ means ‘arrangements’. We use it to refer to the number of ways of arranging a set of objects. In other words, we use permutations when we are concerned about ‘order’.

- **Permutation of n objects**

  an ordered arrangement of the n objects

We often call such a permutation a "**permutation of n objects taken n at a time**," .

$$_nP_n = n \times (n-1) \times (n-2) \times \ldots \times 1 = n!$$

Factorial, written **n!**

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

- **Permutation of n objects taken r at a time**

  ordered arrangement of different objects in positions. The number of such permutations is:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Example 1: Permutation for Orchestra Programs

The school orchestra is planning to play six pieces of music at their next concert. How many different programs are possible?

Solution:

6! = 6*5*4*3*2*1= 720

$$P(6, 6) = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{720}{1} = 720$$

Example 2: **Permutation for Race Cars**

Assume that 10 cars are in a race. In how many ways can three cars finish in first, second and third place?

Solution

$$P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 10\cdot 9\cdot 8 = 720$$

Example 3: **Permutation for Club Officers**

The Volunteer Club has 18 members. An election is held to choose a president, vice-president and secretary. In how many ways can the three officers be chosen?

Solution

The order in which the officers are chosen matters so this is a permutation.

$$P(18,3) = \frac{18!}{(18-3)!} = \frac{18!}{15!} = 18\cdot 7\cdot 6 = 4896$$

3- Combinations

Combinations is a technical term meaning ‘selections’. We use it to refer to the number of different sets of a certain size that can be selected from a larger collection of object where order does not matter.

A **combination** is a selection of objects in which the order of selection does not matter. The number of combinations of n items taking r at a time is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Example 1 : Combination for Picking Books

A student has a summer reading list of eight books. The student must read five of the books before the end of the summer. In how many ways can the student read five of the eight books?

Solution

The order of the books is not important, only which books are read. This is a combination of eight items taking five at a time.

$$C(8,5) = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1} = \frac{8\times7\times6}{3\times2\times1} = 8\times7 = 56$$

Example 2 : Combination for Halloween Candy

A child wants to pick three pieces of Halloween candy to take in her school lunch box. If there are 13 pieces of candy to choose from, how many ways can she pick the three pieces?

Solution

This is a combination because it does not matter in what order the candy is chosen.

$$C(13,3)=\frac{13!}{3!(13-3)!}=\frac{13!}{3!10!}$$
$$=\frac{13\times 12\times 11\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(3\times 2\times 1)(10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}$$
$$=\frac{13\times 12\times 11}{3\times 2\times 1}$$
$$=\frac{1716}{6}=286$$

Example 3: Permutation or Combination for Choosing Men and Women

A class consists of 15 men and 12 women. In how many ways can two men and two women be chosen to participate in an in-class activity?

Solution

This is a combination since the order in which the people is chosen is not important.

Choose two men:

$$C(15,2)=\frac{15!}{2!(15-2)!}=\frac{15!}{2!13!}=105$$

Choose two women:

$$C(12,2)=\frac{12!}{2!(12-2)!}=\frac{12!}{2!10!}=66$$

We want 2 men and 2 women so multiply these results.

105(66)=6930

Example 4: Twelve (12) patients are available for use in a research study. Only seven (7) should be assigned to receive the study treatment. How many different subsets of seven patients can be selected?

Answer

The formula for the number of combinations of 12 objects taken 7 at a time tells us that there are:

$$\binom{12}{7} = \frac{12!}{7!(12-7)!} = 792$$

different possible subsets of 7 patients that can be selected.