Basic definitions and work sheet probability

We often specify a set by listing its members, or **elements**, in parentheses like this $\{\}$.

For example $A = \{2, 4, 6, 8\}$ means that A is the set consisting of numbers 2,4,6,8.

We could also write $A =$\{even numbers less than 9\}.

The **union** of A and B is the set of elements which belong to A or to B (or both) and can be written as $A \cup B$.

The **intersection** of A and B is the set of elements which belong to both A and B, and can be written as $A \cap B$.

The **complement** of A, frequently denoted by $\overline{A}$, is the set of all elements which do not belong to A. In making this definition we assume that all elements we are thinking about belong to some larger set U, which we call the **universal** set.

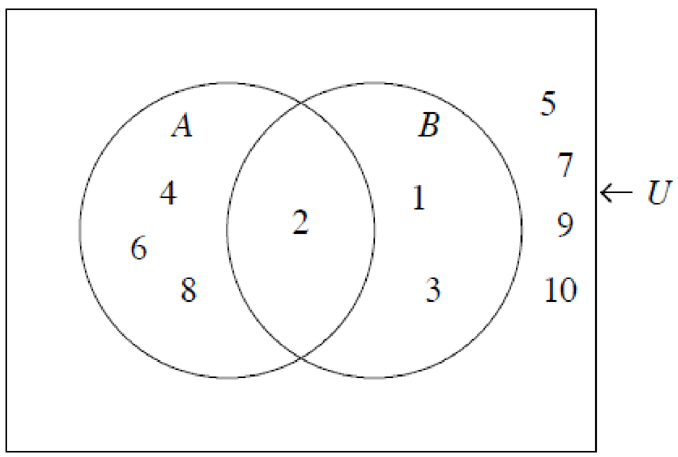
The **empty set**, written $\emptyset$ or $\{\}$, means the set with no elements in it.

A set C is a **subset** of A if **all** the elements in C are also in A.

For example, let

$$\begin{aligned} U &= \{\text{all positive numbers} \leq 10\} \\ A &= \{2, 4, 6, 8\} \\ B &= \{1, 2, 3\} \\ C &= \{6, 8\} \end{aligned}$$

Sets A, B and U may be represented in a Venn Diagram as follows:



A intersection B, $A \cap B$, is shown in the Venn diagram by the overlap of the sets A and B, $A \cap B = \{2\}$.

The union of the sets A and B, $A \cup B$, is the set of elements that are in $A = \{2, 4, 6, 8\}$ together with the elements that are in $B = \{1, 2, 3\}$ including each element once only.

So, $A \cup B = \{1, 2, 3, 4, 6, 8\}$.

The complement of A is the set $\overline{A}$ is contains all the elements in U which are not in A. So, $\overline{A} = \{1, 3, 5, 7, 9, 10\}$.

C is a subset of A as every element in $C = \{6, 8\}$ is also in $A = \{2, 4, 6, 8\}$.

Finite Equiprobable Spaces

In loose terms, we say that the probability of something happening is $\frac{1}{4}$, if, when the experiment is repeated often under the same conditions, the stated result occurs 25% of the time.

For the moment, we will confine our discussion to cases where there are a finite number of equally likely outcomes.

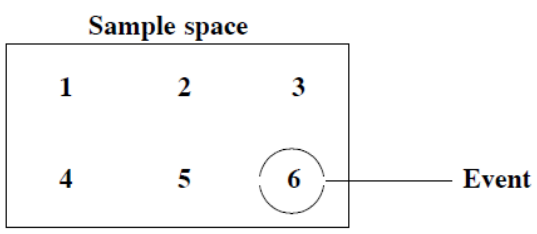
For example, if a coin is tossed, there are two equally likely outcomes: heads (H) or tails (T). If a die is tossed, there are six equally likely outcomes: 1,2,3,4,5,6.

Some Notation

The set of all possible outcomes of the given experiment is called the **sample space**. An **event** is a subset of a sample space.

Calculating Probabilities

Look again at the example of rolling a six faced die. The possible outcomes in this experiment are 1,2,3,4,5,6, so the sample space is the set {1,2,3,4,5,6}. The 'event' of 'getting a 6' is the subset {6}. We represent this in the following diagram.



There are six possibilities in the sample space and only one of these corresponds to getting a 6, so the probability of getting a 6 when you roll a die is $\frac{1}{6}$.
We say that the probability of an event A occuring is

$$P(A) = \frac{\text{Number of elements in } A}{\text{Total number of elements in the sample space}}$$

Example

If a fair coin is tossed, it is clear from our definition of probability above that P (obtaining a head) $= \frac{1}{2}$.

Example

A card is selected at random from a pack of 52 cards. Let $A =$ 'the card is a heart' and $B =$ 'the card is an ace'.
Find $P(A),\ P(B)$.

Solution

$P(A) = \frac{13}{52}$ since there are 13 hearts in the pack. $P(B) = \frac{4}{52}$ since there are 4 aces in the pack.

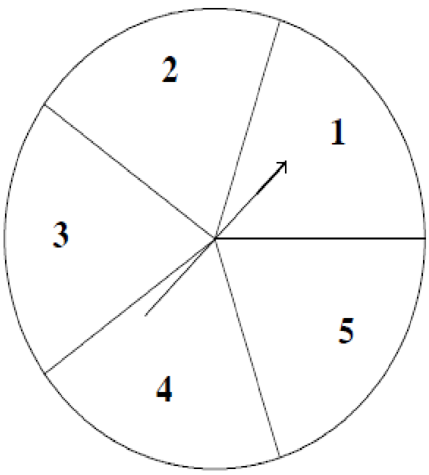
To calculate the probability of an event, we simply need to find out the total number of possible outcomes of an experiment and the number of outcomes which correspond to the **given** event.

Exercise 1

What are your chances of winning a raffle in which 325 tickets have been sold, if you have one ticket?

Exercise 2

A cursor is spun on a disc divided into five equal sectors as shown below. The position of the pointer is noted. (If it is on a line the cursor is spun again.)



Let A be the event 'pointer is in the first sector' and B the event 'pointer is in the 2nd or 4th sector'.
Find $P(A)$, $P(B)$.

Example

Consider the following problem. Two coins are tossed. Let A be the event 'two heads are obtained', and, B be the event 'one head and one tail is obtained'.

Find $P(A),\ P(B)$.

Solution

The sample space=$\{$HH, HT, TH, TT$\}$

A=$\{$HH$\}$

B=$\{$HT, TH$\}$.

Since there are 4 outcomes in the sample space.

$P(A) = \dfrac{1}{4}$

$P(B) = \dfrac{2}{4} = \dfrac{1}{2}$

Notice that HT and TH must be regarded as **different** outcomes.

Often we can conveniently represent the possible outcomes on a diagram and count directly. We will also develop some techniques and rules to assist in our calculations.

Now let us see what happens in reality. Try the following experiment:

Roll a die 50 times and record the number of each of the outcomes 1,2,3,4,5,6.

Continue rolling and record the number of each outcome after 100 rolls. Now record the number after 200 rolls. Find the **relative frequency** of each outcome after 50, 100 and 200 rolls.

For example calculate $\dfrac{\text{the number of times `1' occurs}}{\text{total number of rolls}}$.

Does it get closer to $\frac{1}{6}$, i.e. 0.17?

Complementary Events

If an event is a **certainty**, then its probability is one.

In common language we often say it is 100% certain (which is the same thing).

For example, in the coin tossing experiment, let C be the event 'obtaining a head or a tail'.

The sample space is $\{$H, T$\}$. The event is $\{$H, T$\}$.

So $P(C) = \frac{2}{2} = 1$.

Example

If a normal die is rolled, what is the probability that the number showing is less than 7?

Solution

Sample space = $\{1,2,3,4,5,6\}$

Event = $\{1,2,3,4,5,6\}$

Hence the probability (number is less than 7) = $\frac{6}{6} = 1$.

$\boxed{\text{If an event is impossible, then its probability is zero.}}$

Example

Find the probability of throwing an 8 on a normal die.

Here there are **no** possible outcomes in the event.

i.e. Sample space = $\{1,2,3,4,5,6\}$

Event = $\{\}$, i.e. the empty set.

Hence the probability of throwing an 8 is $\frac{0}{6} = 0$.

If the event is neither impossible nor certain, then clearly its **probability is between 0 and 1**.

Two events are **complementary** if they cannot occur at the same time and they make up the whole sample space.

Example

When a coin is tossed, the sample space is $\{H, T\}$ and the events H = 'obtain a head' and T = 'obtain a tail' are complementary.

If we calculate the probabilities we find that

$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$ and $P(H) + P(T) = 1$.

Example

A die is rolled. Let A be the event 'a number less than 3 is obtained' and let B be the event 'a number of 3 or more is obtained'.

Then $P(A) = \frac{2}{6}$, and $P(B) = \frac{4}{6}$.

So that $P(A) + P(B) = 1$.

We have illustrated following law.

If two events are **complementary**, then their **probabilities add up to 1**.

Example

A marble is drawn at random from a bag containing 3 red, 2 blue, 5 green and 1 yellow marble. What is the probability that it is not green?

Solution

There are two ways of doing this problem.

Method A:

We can work out the probability that the marble is green:

$P(G) = \frac{5}{11}$.

Since a marble is either green or not green, the probability that it is not green,

$P(\overline{G}) = 1 - \frac{5}{11} = \frac{6}{11}$.

Method B:

Alternatively, we can find the probability that the marble is red, blue or yellow which is $\frac{6}{11}$.

Example

What is the probability of drawing a heart or spade from a pack of 52 cards when one card is drawn at random?

Solution

$P(\text{heart}) = \frac{13}{52}$

$P(\text{spade}) = \frac{13}{52}$

$P(\text{heart or spade}) = \frac{26}{52}$ since 26 of the cards are either hearts or spades.

Notice $P(\text{heart or spade}) = P(\text{heart}) + P\ (\text{spade})$.

We may now state the **addition law** for **mutually exclusive events**.

If two events A and B are mutually exclusive, the probability of A or B happening, denoted $P(A \cup B)$, is:

$$P(A \cup B) = P(A) + P(B).$$

Conditional Probability

A lecture on a topic of public health is held and 300 people attend. They are classified in the following way:

Gender	Doctors	Nurses	Total
Female	90	90	180
Male	100	20	120
Total	190	110	300

If one person is selected at random, find the following probabilities:

(a) P(a doctor is chosen);

(b) P(a female is chosen);

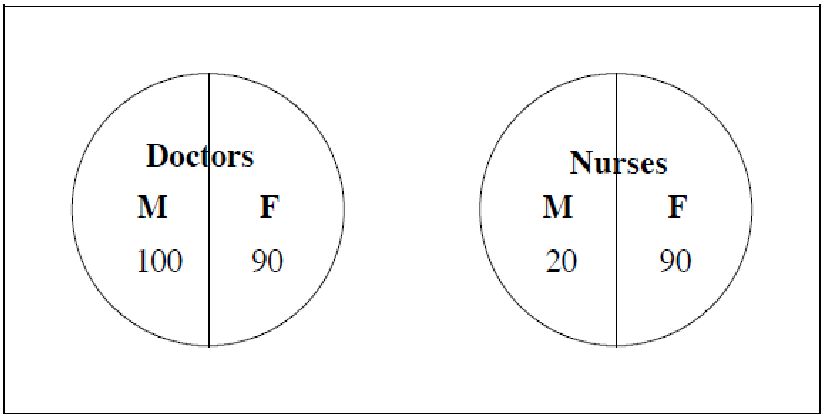
(c) P(a nurse is chosen);

(d) P(a male is chosen);

(e) P(a female nurse is chosen);

(f) P(a male doctor is chosen).

Solution:



(a) The number of doctors is 190 and the total number of people is 300, so $P(\text{doctor}) = \frac{190}{300}$

(b) $P(\text{female}) = \frac{180}{300}$

(c) $P(\text{male}) = \frac{120}{300}$

(d) $P(\text{nurse}) = \frac{110}{300}$

(e) There are 90 female nurses, therefore $P(\text{female} \cap \text{nurse}) = \frac{90}{300}$

(f) $P(\text{male doctor}) = P(\text{male} \cap \text{doctor}) = \frac{100}{300}$.

Now suppose you are given the information that a female is chosen and you wish to find the probability that she is a nurse. This is what we call **conditional probability**. We want the probability that the person chosen is a nurse, subject to the condition that we know she is female. The notation used for this is:

$$P(\text{nurse} \mid \text{female})$$

Read this as 'the probability of the person chosen being a nurse, **given** that she is female'. Since there are 180 females and of these 90 are nurses, the required probability is $\frac{90}{180} = \frac{1}{2}$.

We can see that
$$\begin{aligned} P(\text{nurse} \mid \text{female}) &= \frac{90}{180} \\ &= \frac{90/300}{180/300} \\ &= \frac{P(\text{nurse} \cap \text{female})}{P(\text{female})} \end{aligned}$$

Summary

1. If there are a finite number of equally likely outcomes of an experiment, the probability of an event A is

$$P(A) = \frac{\text{Number of possible outcomes in } A}{\text{Total number of possible outcomes}}$$

2. The probability of an event happening lies between zero and one. If the event cannot happen, its probability is zero and if it is certain to happen, its probability is one.

3. If two events are **complementary**, i.e. they are mutually exclusive (can't happen together) and make up the whole sample space, then their probabilities add up to 1.

4. For two events A and B the probability of A or B (or both) happening is

   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

In particular if A and B are **mutually exclusive**, $P(A \cup B) = P(A) + P(B)$. That is, the chance that **at least one of them** will happen equals the sum of their probabilities.

5. The **conditional probability** of A given B is $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided that $P(B) \neq 0$.

   So $P(A \cap B) = P(A \mid B).P(B)$.

6. A and B are defined to be independent events if $P(A) = P(A \mid B)$. That is, knowing the outcome of one event does not change the probability of the outcome of the other. From (5.) above we see that in this case $P(A \cap B) = P(A).P(B)$, that is the probability of **both** events happening is the **product** of the individual probabilities.