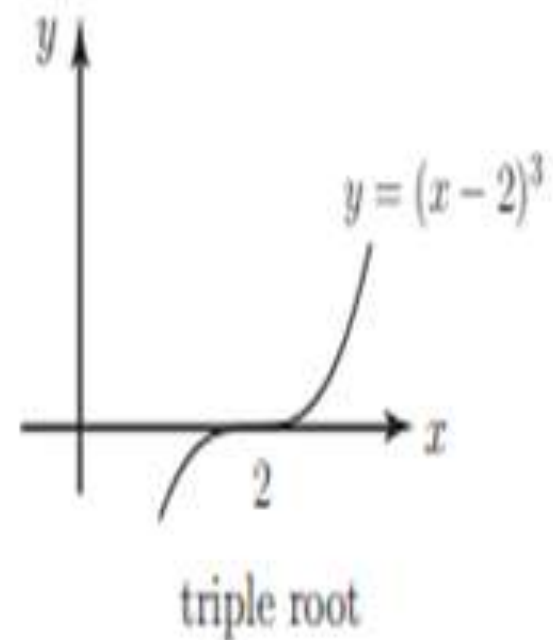
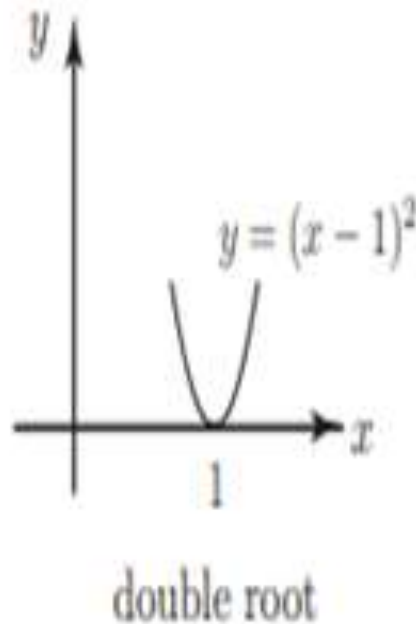
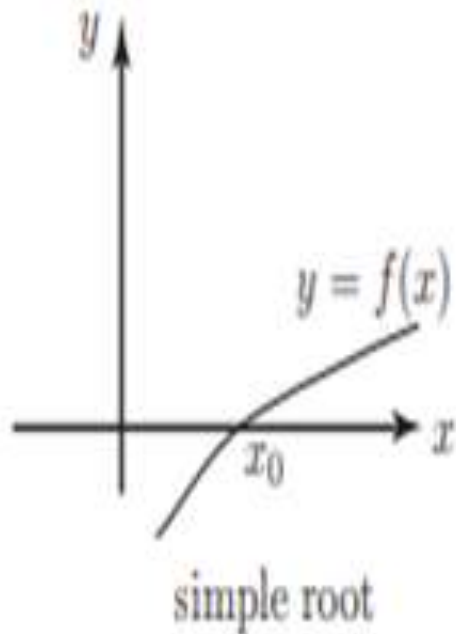


The Newton-Raphson method

We first remind the reader of some basic notation: If $f(x)$ is a given function the value of x for which $f(x) = 0$ is called a **root** of the equation or **zero** of the function. We also distinguish between various types of roots: simple roots and multiple roots. **As per figures below.**



More precisely; a root x_0 is said to be:

a **simple root** if $f(x_0) = 0$ and $\left. \frac{df}{dx} \right|_{x_0} \neq 0$.

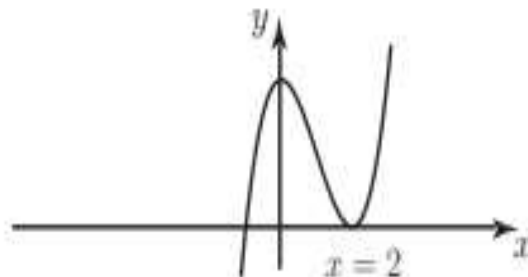
a **double root** if $f(x_0) = 0$, $\left. \frac{df}{dx} \right|_{x_0} = 0$ and $\left. \frac{d^2f}{dx^2} \right|_{x_0} \neq 0$, and so on.

In this Section we shall concentrate on the location of simple roots of a given function $f(x)$.

EXAMPLE -1:

Given graphs of the functions (a) $f(x) = x^3 - 3x^2 + 4$, (b) $f(x) = 1 + \sin x$ classify the roots into simple or multiple.

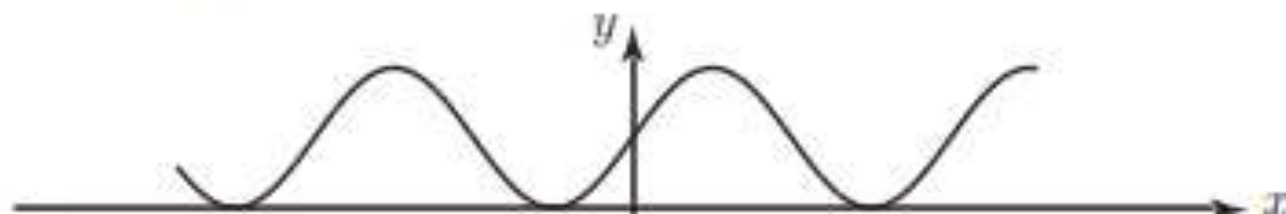
(a) $f(x) = x^3 - 3x^2 + 4$: The negative root is: and the positive root is:



Answer

The negative root is simple and the positive root is double

(b) $f(x) = 1 + \sin x$: Each root is a root



Answer

Each root is a double root.

Finding roots of the equation $f(x) = 0$

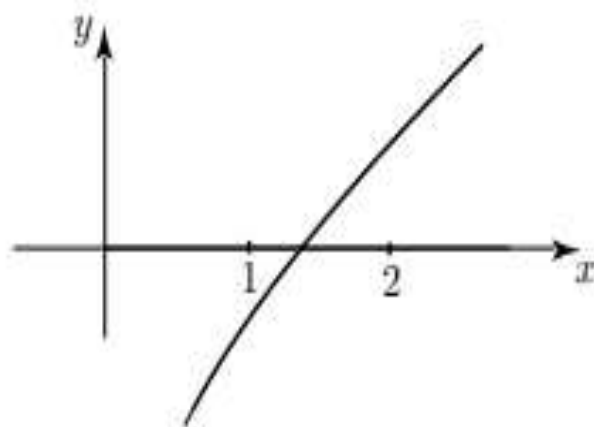
A first investigation into the roots of $f(x)$ might be graphical. Such an analysis will supply information as to the approximate location of the roots.

Sketch the function

$$f(x) = x - 2 + \ln x \quad x > 0$$

and estimate the value of the root.

Answer



A simple root is located near 1.5

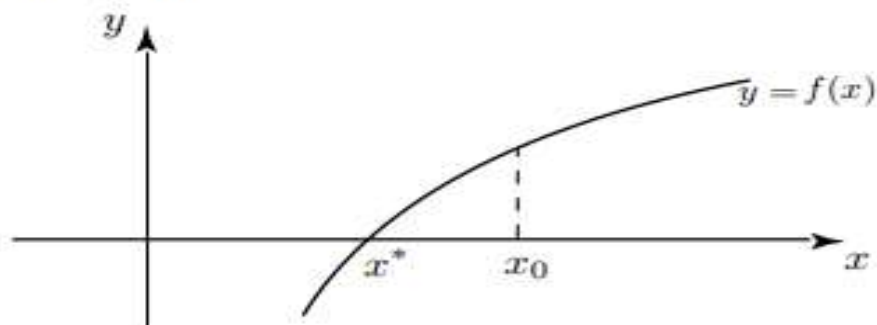
One method of obtaining a better approximation is to halve the interval $1 \leq x \leq 2$ into $1 \leq x \leq 1.5$ and $1.5 \leq x \leq 2$ and test the **sign** of the function at the end-points of these new regions. We find

x	$f(x)$
1	< 0
1.5	< 0
2	> 0

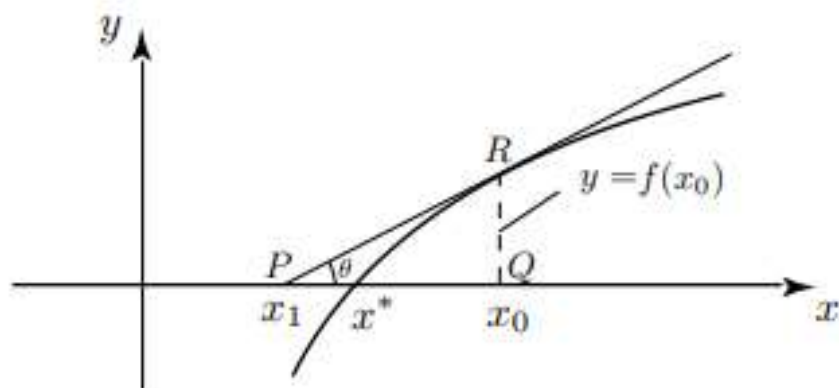
so a root must lie between $x = 1.5$ and $x = 2$ because the sign of $f(x)$ changes between these values and $f(x)$ is a continuous curve. We can repeat this procedure and divide the interval $(1.5, 2)$ into the two new intervals $(1.5, 1.75)$ and $(1.75, 2)$ and test again. This time we find

so a root lies in the interval $(1.5, 1.75)$. It is obvious that proceeding in this way will give a smaller and smaller interval in which the root must lie. But can we do better than this rather laborious bisection procedure? In fact there are many ways to improve this numerical search for the root. In this Section we examine one of the best methods: the **Newton-Raphson** method.

To derive the method we examine the general characteristics of a curve in the neighbourhood of a simple root. Consider Figure 24 showing a function $f(x)$ with a simple root at $x = x^*$ whose value is required. Initial analysis has indicated that the root is approximately located at $x = x_0$. The aim is to provide a better estimate to the location of the root.



The basic premise of the Newton-Raphson method is the assumption that the curve in the close neighbourhood of the simple root at x^* is approximately a straight line. Hence if we draw the tangent to the curve at x_0 , this tangent will intersect the x -axis at a point closer to x^* than is x_0 : see Figure 25.



From the geometry of this diagram we see that

$$x_1 = x_0 - PQ$$

But from the right-angled triangle PQR we have

$$\frac{RQ}{PQ} = \tan \theta = f'(x_0)$$

and so
$$PQ = \frac{RQ}{f'(x_0)} = \frac{f(x_0)}{f'(x_0)} \quad \therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If $f(x)$ has a simple root near x_0 then a closer estimate to the root is x_1 where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

conclusion

If $f(x)$ has a simple root near x_n then a closer estimate to the root is x_{n+1} where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the **Newton-Raphson iterative formula**. The iteration is begun with an initial estimate of the root, x_0 , and continued to find x_1, x_2, \dots until a suitably accurate estimate of the position of the root is obtained. This is judged by the convergence of x_1, x_2, \dots to a fixed value.

Example

$f(x) = x - 2 + \ln x$ has a root near $x = 1.5$. Use the Newton-Raphson method to obtain a better estimate.

Solution

Here $x_0 = 1.5$, $f(1.5) = -0.5 + \ln(1.5) = -0.0945$

$$f'(x) = 1 + \frac{1}{x} \quad \therefore \quad f'(1.5) = 1 + \frac{1}{1.5} = \frac{5}{3}$$

Hence using the formula:

$$x_1 = 1.5 - \frac{(-0.0945)}{(1.6667)} = 1.5567$$

The Newton-Raphson formula can be used again: this time beginning with 1.5567 as our estimate:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5567 - \frac{f(1.5567)}{f'(1.5567)} = 1.5567 - \frac{\{1.5567 - 2 + \ln(1.5567)\}}{\left\{1 + \frac{1}{1.5567}\right\}} \\ &= 1.5567 - \frac{\{-0.0007\}}{\{1.6424\}} = 1.5571 \end{aligned}$$

This is in fact the correct value of the root to 4 d.p., which calculating x_3 would confirm.

EXAMPLE 2:

The function $f(x) = x - \tan x$ has a simple root near $x = 4.5$. Use one iteration of the Newton-Raphson method to find a more accurate value for the root.

Solution:

First find df/dx

$$\frac{df}{dx} = 1 - \sec^2 x = -\tan^2 x$$

Now use the formula $x_1 = x_0 - f(x_0)/f'(x_0)$ with $x_0 = 4.5$ to obtain x_1

$$f(4.5) = 4.5 - \tan(4.5) =$$

$$f'(4.5) = 1 - \sec^2(4.5) = -\tan^2(4.5) =$$

$$x_1 = 4.5 - \frac{f(4.5)}{f'(4.5)} =$$

$$f(4.5) = -0.1373, \quad f'(4.5) = -21.5048$$

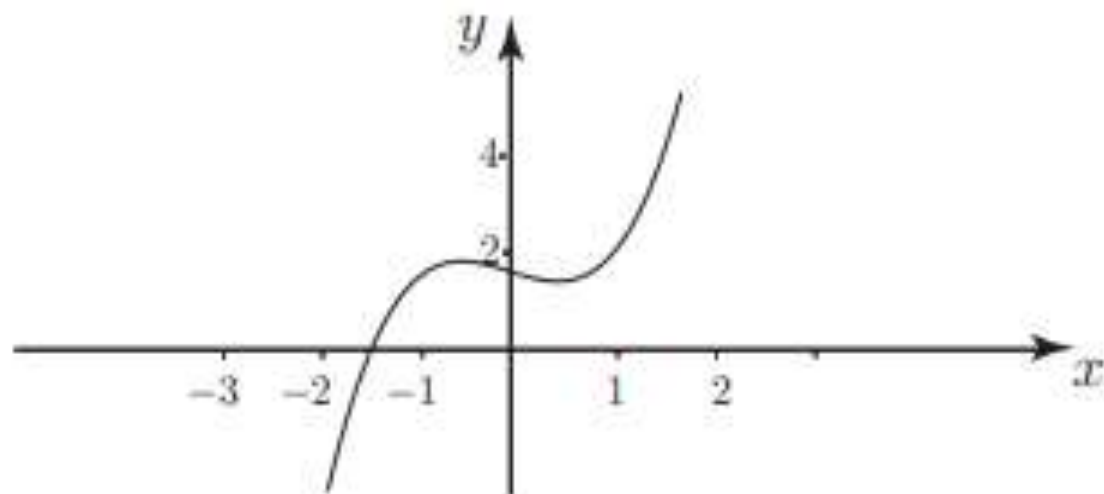
$$\therefore x_1 = 4.5 - \frac{0.1373}{21.5048} = 4.4936.$$

As the value of x_1 has changed little from $x_0 = 4.5$ we can expect the root to be 4.49 to 3 d.p.

EXAMPLE -3

Sketch the function $f(x) = x^3 - x + 3$ and confirm that there is a simple root between $x = -2$ and $x = -1$. Use $x_0 = -2$ as an initial estimate to obtain the value to 2 d.p.

First sketch $f(x) = x^3 - x + 3$ and identify a root:



Clearly a simple root lies between $x = -2$ and $x = -1$.

Now use one iteration of Newton-Raphson to improve the estimate of the root using $x_0 = -2$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x) = x^3 - x + 3, \quad f'(x) = 3x^2 - 1 \quad x_0 = -2$$

$$\therefore x_1 = -2 - \frac{\{-8 + 2 + 3\}}{11} = -2 + \frac{3}{11} = -1.727$$

Now repeat this process for a second iteration using $x_1 = -1.727$:

$$x_2 = x_1 - f(x_1)/f'(x_1)$$

$$\begin{aligned} x_2 &= -1.727 - \{-(1.727)^3 + 1.727 + 3\}/\{3(1.727)^2 - 1\} \\ &= -1.727 + \{(0.424)/(7.948)\} = -1.674 \end{aligned}$$

Repeat for a third iteration and state the root to 2 d.p

$$x_3 = x_2 - f(x_2)/f'(x_2)$$

$$\begin{aligned}x_3 &= -1.674 - \{-(1.674)^3 + 1.674 + 3\}/\{3(1.674)^2 - 1\} \\ &= -1.674 + \{0.017\}/\{7.407\} = -1.672\end{aligned}$$

We conclude the value of the simple root is -1.67 correct to 2 d.p.

Exercises

1. By sketching the function $f(x) = x - 1 - \sin x$ show that there is a simple root near $x = 2$. Use two iterations of the Newton-Raphson method to obtain a better estimate of the root.
2. Obtain an estimation accurate to 2 d.p. of the point of intersection of the curves $y = x - 1$ and $y = \cos x$.

Answers

1. $x_0 = 2, \quad x_1 = 1.936, \quad x_2 = 1.935$

2. The curves intersect when $x - 1 - \cos x = 0$. Solve this using the Newton-Raphson method with initial estimate (say) $x_0 = 1.2$.

The point of intersection is $(1.28342, 0.283437)$ to 6 significant figures.