Bayes' Theorem

Bayes' Theorem, named after 18th-century British mathematician Thomas Bayes, is a mathematical formula for determining conditional probability. Conditional probability is the likelihood of an outcome occurring, based on a previous outcome having occurred in similar circumstances.

the Bayes' theorem describes the probability of an event based on prior knowledge of the conditions that might be relevant to the event .The theorem has become a useful element in the implementation of machine learning. Besides statistics, the Bayes' theorem is also used in various disciplines, with medicine and pharmacology as the most notable examples. In addition, the theorem is commonly employed in different fields of finance. Some of the applications include but are not limited to, modeling the risk of lending money to borrowers or forecasting the probability of the success of an investment.

❖ Conditional probability

Given events A and B , often we are interested in statements like
if even A has occurred, then the probability of B is ...
We used P(B|A) to denoted the conditional probability of event B occurring, given that event A has already occurred.
The following formula was provided for finding P(B|A):

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example 1: **Roll a die**

Let $A = \{score\ an\ even\ number\}$ and $B = \{score\ a\ number\ \geq\ 3\}$.

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{2}{3}, \quad P(A \cap B) = \frac{1}{3}$$

because the intersection has only two elements, then

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

$$P(A/B) \neq P(B/A)$$

Example 2 : **Roll a die**

Let $A = \{one, two, three\}$ and $B = \{two, four\}$. Are A and B independent?

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{3}, \quad P(A \cap B) = \frac{1}{6}$$

then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/3} = \frac{1}{2} = P(A)$$

$$P(B|A) = \frac{P \cap A}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3} = P(B)$$

Thus we conclude that A and B are independent.

❖ Formula for Bayes' Theorem

The probability of event A, given that event B has subsequently occurred, is

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{[P(A) \cdot P(B|A)] + [P(\overline{A}) \cdot P(B|\overline{A})]}$$

Where:

- P(A|B) – the probability of event A occurring, given event B has occurred
- P(B|A) – the probability of event B occurring, given event A has occurred
- P(A) – the probability of event A
- P(B) – the probability of event B

Note that events A and B are independent events (i.e., the probability of the outcome of event A does not depend on the probability of the outcome of event B).

Example 3:

In Orange County, 51% of the adults are males. (It doesn't take too much advanced mathematics to deduce that the other 49% are females.) One adult is randomly selected for a survey involving credit card usage.

a. Find the prior probability that the selected person is a male.

b. It is later learned that the selected survey subject was smoking a cigar. Also, 9.5% of males smoke cigars, whereas 1.7% of females smoke cigars (based on data from the Substance Abuse and Mental Health Services Administration). Use this additional information to find the probability that the selected subject is a male.

Solution

Let's use the following notation:

M = male M^{-} = female (or not male)

C = cigar smoker C^{-} = not a cigar smoker.

a. Before using the information given in part b, we know only that 51% of the adults in Orange County are males, so the probability of randomly selecting an adult and getting a male is given by P(M) = 0.51.

b. Based on the additional given information, we have the following:

P(M) = 0.51 because 51% of the adults are males

P(M¯) = 0.49 because 49% of the adults are females (not males)

P(C|M) = 0.095 because 9.5% of the males smoke cigars (That is، the probability of getting someone who smokes cigars, given that the person is a male, is 0.095).

P(C|M¯) = 0.017. because 1.7% of the females smoke cigars (That is، the probability of getting someone who smokes cigars, given that the person is a female, is 0.017).

Let's now apply Bayes' theorem by using the preceding formula with M in place of A, and C in place of B. We get the following result:

$$P(M \mid C) = \frac{P(M) \cdot P(C|M)}{[P(M) \cdot P(C|M)] + [P(\overline{M}) \cdot P(C|\overline{M})]}$$

$$= \frac{0.51 \cdot 0.095}{[0.51 \cdot 0.095] + [0.49 \cdot 0.017]}$$

$$= 0.85329341$$

$$= 0.853 \text{ (rounded)}$$

Example 4:

An aircraft emergency locator transmitter (ELT) is a device designed to transmit a signal in the case of a crash. The Altigauge Manufacturing Company makes 80% of the ELTs، the Bryant Company makes 15% of them, and the Chartair Company makes the other 5%.

The ELTs made by Altigauge have a 4% rate of defects, the Bryant ELTs have a 6% rate of defects, and the Chartair ELTs have a 9% rate of defects (which helps to explain why Chartair has the lowest market share).

a. If an ELT is randomly selected from the general population of all ELTs, find the probability that it was made by the Altigauge Manufacturing Company.

b. If a randomly selected ELT is then tested and is found to be defective, find the probability that it was made by the Altigauge Manufacturing Company.

Solution

We use the following notation:

A = ELT manufactured by Altigauge

B = ELT manufactured by Bryant

C = ELT manufactured by Chartair

D = ELT is defective

D^- = ELT is not defective (or it is good)

a. If an ELT is randomly selected from the general population of all ELTs, the probability that it was made by Altigauge is 0.8 (because Altigauge manufactures 80% of them)

b. If we now have the additional information that the ELT was tested and was found to be defective, we want to revise the probability from part (a) so that the new information can be used. We want to find the value of $P(A|D)$, which is the probability that the ELT was made by the Altigauge company given that it is defective. Based on the given information, we know these probabilities:

$P(\text{A}) = 0.80$ because Altigauge makes 80% of the ELTs
$P(\text{B}) = 0.15$ because Bryant makes 15% of the ELTs
$P(\text{C}) = 0.05$ because Chartair makes 5% of the ELTs

$P(\text{D}|\text{A}) = 0.04$ because 4% of the Altigauge ELTs are defective
$P(\text{D}|\text{B}) = 0.06$ because 6% of the Bryant ELTs are defective
$P(\text{D}|\text{C}) = 0.09$ because 9% of the Chartair ELTs are defective

Here is Bayes' theorem extended to include three events corresponding to the selection of ELTs from the three manufacturers (A, B, C):

$$P(A|D) = \frac{P(A)\cdot P(D|A)}{[P(A)\cdot P(D|A)]+[P(B)\cdot P(D|B)]+[P(C)\cdot P(D|C)]}$$

$$= \frac{0.80\cdot 0.04}{[0.80\cdot 0.04]+[0.15\cdot 0.06]+[0.05\cdot 0.09]}$$

$$= 0.703 \text{ (rounded)}$$