Theorems of Probability

You must have heard in news channel the reporter saying that there are some chances of rain tomorrow. Your friend, after an exam, says that he will score 90% in the exams Also, some news **reports** say one out of five children are suffering from malnutrition. What are all these? How can they predict and calculate all these? These are some of the examples of **Probability**. What is probability? Let us get familiar with it.

What is Probability?

The words like 'certain', 'maybe', 'probably', 'never' are related to the term probability. It is a measure for calculating the chances or the possibilities of the occurrence of **a random event** . In simple words, it calculates the chance of the favorable outcome amongst the entire possible outcome.

Mathematically, if you want to answer what is probability, it is defined as the ratio of the number of favorable events to the total number of possible outcomes of **a random experiment**. It is denoted by 'p'. The probability of an event, say, E,

$$Probability(P) = \frac{Favourable\ Outcomes}{Total\ Outcomes}$$

$$P(A) = \frac{n(A)}{n(S)}$$

Where,

P(A) represents the probability of event A,
n(A) represents the number of favorable outcomes, and
n(S) represents total outcomes in sample space.

Calculating Probability

In an experiment, the probability of an event is the possibility of that event occurring. The probability of any event is a value between (and including) "0" and "1". Follow the steps below for calculating probability of an event A:

- **Step 1:** Find the sample space of the experiment and count the elements. Denote it by n(S).
- **Step 2:** Find the number of favorable outcomes and denote it by n(A).
- **Step 3:** To find probability, divide n(A) by n(S). i.e., P(A) = n(A)/n(S).

Terminologies associated with probability

1. Event: Each outcome of an experiment or trial that is conducted is termed an event.

2. Sample Space: The set of all the possible outcomes of an experiment is called the sample space of that experiment and is generally represented as S.

3. Sure Event: An event that will always occur is called a sure event. A sure event has a probability of 1.

4. Impossible Event: An event that will never occur is called an impossible event. An impossible event has a probability of 0.

5. Favorable Outcome: An event that produces the desired result in an experiment is called a favorable outcome for that experiment.

6. **Random Experiment:** Those experiments, (conducted in identical situations), in which the outcomes are the same in each trail but its prediction is uncertain. The experiment in which the total numbers of outcomes are the same in all cases but the occurrence of the outcome is unpredictable is random.

7. **Outcome:** The various possible results of a random experiment.

Basic Features of Probability

- The probability ranges from 0 to 1. $0 \leq P(A) \leq 1$
- a sure event $P(A) = 1$; impossible event $P(\phi) = 0$
- P[sum of all possible events]=1.

$$P(A)+P(B)+\ldots+P(N)=1$$

- P[sum of events]= Sum of probabilities of events.

Example : A dice is rolled. What is the probability that an even number has been obtained?

Probability of Rolling a Dice

Answer:

When fair six-sided dice are rolled, there are six possible outcomes: 1, 2, 3, 4, 5, or 6.

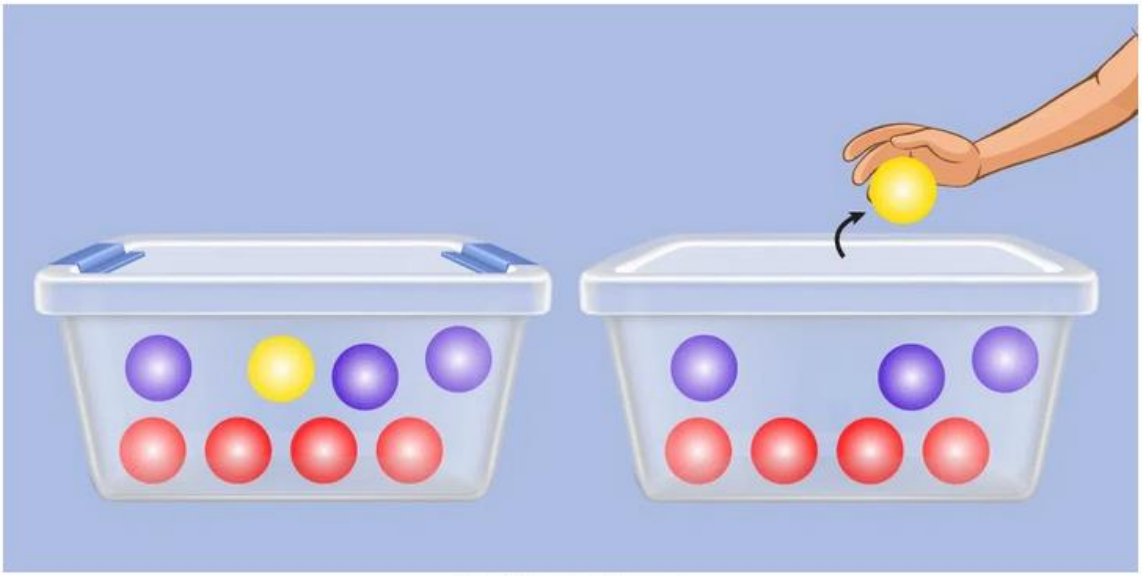
Out of these, half are even (2, 4, 6) and half are odd (1, 3, 5). Therefore, the probability of getting an even number is:

P(even) = number of even outcomes / total number of outcomes

P(even) = 3 / 6

P(even) = 1/2

Example . There are 8 balls in a container, 4 are red, 1 is yellow and 3 are blue. What is the probability of picking a yellow ball?



Probability of picking a ball

Answer:

The probability is equal to the number of yellow balls in the container divided by the total number of balls in the container, i.e. 1/8.

Probability Theorems

I. Theorem of Complementary Events

Two events are said to be complementary events if the sum of their probability is 1. Thus, if A is an event and the probability of A is given by P(A) then this theorem states that

$P(A') = 1 - P(A)$

where P(A’) is the probability of the complementary event of A; i.e., A’. In such cases, events A and A’ are said to be mutually exhaustive also.

Example:

Consider an event A that 3 will appear on rolling a dice. Calculate the probability of not getting a 3.

Solution:

Probability of getting a 3 on dice = $P(A) = \frac{1}{6}$

The probability of A' which is the probability of not getting a 3 is calculated using the theorem of complementary events as follows:

P(A') = 1 – P(A)

$P(A') = 1 - \frac{1}{6}$

$P(A') = \frac{5}{6}$

II. Theorem of Addition

Theorem of Addition is used when one has to determine the probability of occurrence of two or more events. In simple terms, this theorem is used to calculate the probability of union of two or more than two events. **For instance,** there are two events E_1 and E_2 of a given sample space. By using the theorem of addition, we can determine the probability that either **E_1 or E_2** will occur. However, to determine the probability, first of all, we have to find out whether the events are mutually exclusive or overlapping, after that only the required probability is calculated using the correct rule or formula.

Theorem of Addition has two cases:

1. When the events are Mutually Exclusive

If A and B are mutually exclusive events then according to this theorem:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

where, $(E_1 \cup E_2)$ means either E_1 or E_2

If there are more than two events, then

$$P(E_1 \cup E_2 \cup \ldots\ldots E_n) = P(E_1) + P(E_2) + \ldots\ldots + P(E_n)$$

If these events are collectively exhaustive, then

$P(E_1 \cup E_2 \cup \ldots\ldots E_n) = P(E_1) + P(E_2) + \ldots\ldots + P(E_n) = 1$

This rule is known as the Theorem of Addition for Mutually Exclusive Events.

Example:

If A and B are events of occurrence of 2 and occurrence of 3 on a dice respectively, then calculate the probability of occurrence of A or B; i.e., getting 2 or 3 on dice.

Solution:

$Probability\ of\ getting\ 2 = P(A) = \frac{1}{6}$

$Probability\ of\ getting\ 3 = P(B) = \frac{1}{6}$

As A and B are independent and mutually exclusive events, using the theorem of addition,

$P(A \cup B) = P(A) + P(B)$

$P(A \cup B) = \frac{1}{6} + \frac{1}{6}$

$P(A \cup B) = \frac{1}{3}$

2. When the Events are Overlapping

When the events are overlapping, the theorem of addition determines the probability that one or more events would occur in a single trial.

If E_1 and E_2 are overlapping events then according to this theorem:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

where, $P(E_1 \cup E_2)$ is the probability that either E_1 or E_2 or both events will occur, and $P(E_1 \cap E_2)$ is the joint probability that indicates the probability of occurrence of both E_1 and E_2.

Example:

Consider a class with 20 students. 10 students passed in Maths, 15 in English, and 13 in both. What is the probability that a student passed in either Math or English?

Solution:

Total students = 20

P(A) = Probability that a student passed in Maths = $\frac{10}{20}$

P(B) = Probability that a student passed in English = $\frac{15}{20}$

P(A∩B) = Probability that a student passed in both Math and English = $\frac{13}{20}$

This is a case of overlapping events as some students who passed in Math may have passed in English too and some students who passed in English may have passed in Math too. Thus we need to remove these common students from the sum of students who passed in Math and English. Thus, using the theorem of addition for overlapping events, we get:

The probability that a student passed either in Maths or in English = P(A∪B)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{10}{20} + \frac{15}{20} - \frac{13}{20}$

$P(A \cup B) = \frac{12}{20}$

$P(A \cup B) = \frac{3}{5}$

III. Theorem of Multiplication

There are 2 cases in theorem of multiplication:

1. When events are Independent

If E_1 and E_2 are independent events with $P(E_1) \neq 0$ and $P(E_2) \neq 0$, then according to this theorem of probability:

$$P(E_1 \cap E_2) = P(E_1).P(E_2)$$

This means that the probability of intersection of two events E_1 and E_2 is equal to the product of the individual probabilities of events E_1 and E_2.

Example:

A statistics problem is given to two students, say A and B. Their chances of solving it correctly are known to be 0.5 and 0.3, respectively. Find the probability that both of them solve it.

Solution:

Let E_1 be the event of A solving the problem and E_2 is the event of B solving the problem.

Here event E_1 and E_2 are independent.

$$P(E_1) = 0.5$$

$$P(E_2) = 0.3$$

$$P(E_1E_2) = P(E_1)P(E_2)$$

$$P(E_1E_2) = 0.5 \times 0.3$$

$$P(E_1E_2) = 0.15$$

Therefore, the chances of both A and B solving the problem is 15%.

2. When Events are not Independent

If the events are not independent, then multiplication theorem states that the joint probability of the events E_1 and E_2 is given by the probability of event E_1 multiplied by the probability of event E_2 given that event E_1 has occurred and vice-versa. Simply put, the rule uses the concept of conditional probability when the events are known to be dependent or non-independent. According to this theorem, if E_1 and E_2 are two events where $P(E_1) \neq 0$ and $P(E_2) \neq 0$, and if E_1 and E_2 are not independent events, then:

$$P(E_1 \cap E_2) = P(E_1).P(E_2/E_1)$$

$$P(E_1 \cap E_2) = P(E_2).P(E_1/E_2)$$

Example:

A large company employs 70 engineers, of whom 36 are males and the remaining are females. Of the female engineers. 14 are under 35 years of age, 15 are between 35 and 45 years of age, and the remaining are over 45 years of age. What is the probability of randomly selected engineer who is a female and under the age of 35 years of age?

Solution:

Let E_1 represent the event that an engineer selected at random is a female and E_2 is the event that a female engineer selected is under 35 years of age.

Since there are 36 males out of 70 engineers, it means that the number of female engineers is 34.

$P(E_1) = \frac{Total\ Female\ Engineers}{Total\ Engineers}$

$P(E_1) = \frac{34}{70}$

$P(\frac{E_2}{E_1}) = \frac{14}{34}$

Therefore, $P(E_1 \cap E_2) = P(E_1) \times P(\frac{E_2}{E_1})$

$P(E1 \cap E2) = \frac{34}{70} \times \frac{14}{34}$

$P(E1 \cap E2) = \frac{1}{5}$