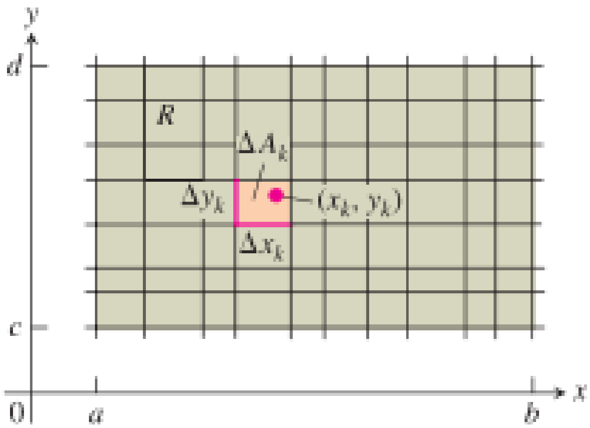
Mathematics II chapter four

4.1 Double Integration (Area, moment, centroid, moment of inertia & volume)

-Area

$$A=\iint dx\,dy \quad or \quad A=\iint dy\,dx$$



Example //Find area enclosed by x=3, x=1, y=0,y=2. using double integration.

Solution/ /

$$A=\iint dx\,dy$$

$$A=\int_0^2\int_1^3 dx\,dy=\int_0^2 (x)\big|_1^3\,dy=\int_0^2 2\,dy=(2y)\big|_0^2=4$$

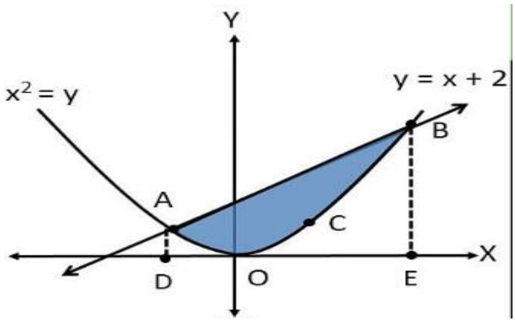
Or

$$A=\iint dy\,dx$$

A=$\int_{1}^{3}\int_{0}^{2} dy\, dx = \int_{1}^{3}(y)|_{0}^{2}\, dx = \int_{1}^{3} 2\, dx = (2x)|_{1}^{3} = 4$

Example //Find the area enclosed by y=x+2 & .
y=x^2

Solution //



$y_1=y_2$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

(X+1) (x-2) =0

X=-1, x=2

A=$\iint dy\, dx$

A=$\int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx = \int_{-1}^{2}(y)|_{x^2}^{x+2}\, dx = \int_{-1}^{2}(x+2) - x^2)\, dx =$
$\left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Big|_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5$

Example//Evaluate $\int_{0}^{\pi}\int_{x}^{\pi} \frac{\sin y}{y} dy\, dx$

Solution // $x_1=0$, $x_2= \pi . y1 = \pi . y2 = x$

$$\int_0^{\pi}\int_0^{y}\frac{\sin y}{y}\,dx\,dy=\int_0^{\pi}\left(\frac{\sin y}{y}x\right)\Big|_0^y dy$$

$$=\int_0^{\pi}\sin y\;dy=(-\cos x)\Big|_0^{\pi}=2$$

Application of double integration

1-volume $=\iint zdA=\iint z\,dx\,dy=\iint z\,dy\,dx$

2-moment

Mx$=\iint ydA=\iint y\,dx\,dy=\iint y\,dy\,dx$

My$=\iint xdA=\iint x\,dx\,dy=\iint x\,dy\,dx$

3-Center of area

$\bar{X}=\frac{My}{A}.\quad \bar{Y}=\frac{Mx}{A}$

$$\left[where\ A=\iint dA=\iint dx\,dy=\iint dy\,dx\right]$$

4. Moment of Inertia

Ix$=\iint y^2dA\quad .Iy=\iint x^2dA.Ir=\iint(x^2+y^2)dA$

Example// Find the centroid enclosed function

Y=6x-x^2 . $y = x$

Solution /

$$y_1=y_2$$

$$6x - x^2 = x$$

$$x^2 - 6x + x = 0$$

$$x^2 - 5x = 0$$

$$X\,(x-5) = 0$$

$$X=0 \text{ or } x=5$$

$$\bar{X} = \frac{My}{A}.\quad \bar{Y} = \frac{Mx}{A}$$

A=$\iint dy\,dx = \int_0^5 \int_x^{6x-x^2} dy\,dx = \int_0^5 (y)|_x^{6x-x^2}\,dx = \int_0^5 (6x - x^2 - x)\,dx = 125/6$

Mx=$\iint y dA = \iint y\,dy\,dx$

Mx=$\int_0^5 \int_x^{6x-x^2} ydy\,dx = \int_0^5 \left(\frac{y^2}{2}\right)\Big|_x^{6x-x^2} dx = \int_0^5 \left(\frac{(6x-x^2)^2}{2} - \frac{(x)^2}{2}\right) dx = 104.16$

Also My=625/12

$$\bar{X} = \frac{My}{A} = \frac{\frac{625}{12}}{\frac{125}{6}} = 2.5$$

$$\bar{Y} = \frac{Mx}{A} = \frac{104.16}{125/6} = 5$$

Example//Calculate second moment of inertia about y-axis for area enclosed by y=x^2 , y=x+2.

Solution //

$$y_1 = y_2$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

X=2 or x=-1

$$Iy = \iint x^2 dA$$

$$= \iint x^2 dy\,dx = \int_{-1}^{2} \int_{x^2}^{x+2} x^2 dy\,dx =$$

$$= \int_{-1}^{2} (x^2 y)\Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} x^2(x + 2 - x^2)dx = \frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5}\Big|_{-1}^{2}$$

$$= 365\ unit\ of\ length$$

Example// Find the volume enclosed by two surfaces

$$z_1 = 2 + x^2 \quad + y^2 \qquad , z_2 = 4 - x^2 - y^2$$

Solution //

$$z_1 = z_2$$

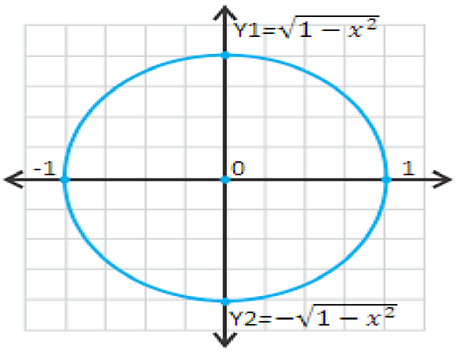
$$2 + x^2 + y^2 = 4 - x^2 - y^2$$

$$2 + x^2 + y^2 - 4 + x^2 + y^2 = 0$$

$$2x^2 + 2y^2 - 2 = 0$$

$$x^2 + y^2 = 1 \leftrightarrow x^2 + y^2 = r^2$$

$$y = \sqrt{1 - x^2}$$



$$\text{Volume} = \iint z\,dA = \iint z\,dy\,dx = \iint (Z2 - Z1)\,dy\,dx = \iint (4 - x^2 - y^2 - 2 - x^2 - y^2)\,dy\,dx$$

$$\text{Volume} = 2\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (2 - 2x^2 - 2y^2)\,dy\,dx$$

Or Volume$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2-2x^2-2y^2)dy\,dx$

Or Volume$=4\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}(2-2x^2-2y^2)dy\,dx$

v$=4\int_{0}^{1}\left(2y-2x^2y-\frac{2y^3}{3}\right)\Big|_{0}^{\sqrt{1-x^2}}dx$

$$=4\int_{0}^{1}\left(2\sqrt{1-x^2}-2x^2(1-x^2)-\frac{2(\sqrt{1-x^2})^3}{3}\right)dx$$