Non-homogenous S.O.D.E

If $y'' + f(x)\,y' + g(x)\,y = r(x)$

The general solution is: $y = y_c + y_p$

y_c : complementary solution

y_p : particular solution

Term in $r(x)$	Choice of y_p
$e^{r(x)}$	$A e^{rx}$
x^n	$A x^n + B x^{n-1} + \cdots$
$\sin kx$, $\cos kx$	$A \cos kx + B \sin kx$

Ex.(1) If $r(x)$ is:

1. $3x^2 \longrightarrow Ax^2 + Bx + c$

2. $x^2 - 4x^4 \longrightarrow Ax^4 + Bx^3 + Cx^2 + Dx + E$

3. $2e^{3x} \longrightarrow Ae^{3x}$

4. $2e^{3x} + x^2 \longrightarrow Ae^{3x} + (Bx^2 + Cx + D)$

5. $5\cos 2x \; ; \; 5\sin 2x \; ; \; 5\cos 2x - 2\sin 2x$

$\longrightarrow A\cos 2x + B\sin 2x$

6. $3\sin 2x - 4\cos x$

$\longrightarrow A\sin 2x + B\cos 2x + C\cos x + D\sin x$

7. $x^2 \sin 2x$

$\longrightarrow (Ax^2 + Bx + C)\sin 2x + (Dx^2 + Ex + f)\cos 2x$

8. $x^2 e^{2x} \longrightarrow (Ax^2 + Bx + C)e^{2x}$

9. $2e^{3x} + 4e^{5x} \longrightarrow Ae^{3x} + Be^{5x}$

EX. 2/ solve $y'' + 4y' + 3y = 5e^{2x}$

sol:

$$m^2 + 4m + 3 = 0$$

$$(m+3)(m+1) = 0$$

$$m_1 = -3 \quad \text{and} \quad m_2 = -1$$

$$\therefore y_c = C_1 e^{-3x} + C_2 e^{-x}$$

$$\left.\begin{aligned} y_p &= A e^{2x} \\ y'_p &= 2A e^{2x} \\ y''_p &= 4A e^{2x} \end{aligned}\right\} \text{sub in D.E}$$

$$4Ae^{2x} + 8Ae^{2x} + 3Ae^{2x} = 5e^{2x}$$

$$15A\cancel{e^{2x}} = 5\cancel{e^{2x}} \longrightarrow A = \frac{1}{3}$$

$$\therefore y_p = \tfrac{1}{3} e^{2x}$$

$$\Rightarrow y(x) = y_c + y_p$$

$$= C_1 e^{-3x} + C_2 e^{-x} + \frac{1}{3} e^{2x}$$

EX(3) Solve $y'' + 4y' + 3y = 5\sin 2x$?

Sol

$m^2 + 4m + 3 = 0 \rightarrow (m+3)(m+1) = 0$

$m_1 = -3$ and $m_2 = -1$

$y_c = C_1 e^{-3x} + C_2 e^{-x}$

$y_p = A\cos 2x + B\sin 2x$

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

$-4A\cos 2x - 4B\sin 2x - 8A\sin 2x + 8B\cos 2x + 3A\cos 2x$
$+ 3B\sin 2x = 5\sin 2x$

$(8B + 3A - 4A)\cos 2x + (-4B - 8A + 3B)\sin 2x = 5\sin 2x$

$-A + 8B = 0 \rightarrow A = 8B$ ---- (1)
$-8A - B = 5 \rightarrow B = -\frac{1}{13}$ and $A = \frac{-8}{13}$

$\therefore y_p = \frac{-8}{13}\cos 2x - \frac{1}{13}\sin 2x$

$y(x) = C_1 e^{-3x} + C_2 e^{-x} - \frac{8}{13}\cos 2x - \frac{1}{3}\sin 2x$

EX(4): Solve $y'' + 9y = 2x^2 + 4x + 7$?

Sol:

$$m^2 + 9 = 0 \longrightarrow m_1 = m_2 = \mp 3i$$

$$y_c = e^{0x}\left[C_1 \cos 3x + C_2 \sin 3x\right]$$

$$\left.\begin{aligned} y_p &= A_0 x^2 + A_1 x + A_2 \\ y'_p &= 2A_0 x + A_1 \\ y''_p &= 2A_0 \end{aligned}\right\} \text{Sub in D.E}$$

$$2A_0 + 9A_0 x^2 + 9A_1 x + 9A_2 = 2x^2 + 4x + 7$$

$$9A_0 = 2 \longrightarrow A_0 = \frac{2}{9}$$

$$9A_1 = 4 \longrightarrow A_1 = 4/9$$

$$2A_0 + 9A_2 = 7 \longrightarrow A_2 = 59/81$$

$$\therefore y_p = 2/9\, x^2 + 4/9\, x + 59/81$$

$$\Rightarrow y(x) = y_c + y_p$$

$$y(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{2}{9} x^2 + 4/9\, x + 59/81$$