Lecture Nine

AC Power Analysis

9.1 Introduction

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. Our major concern in this lecture is power analysis. Power analysis is of paramount importance. Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another. Also, every industrial and household electrical device, such as: fan, motor, lamp, pressing iron, TV, personal computer, has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance.

9.2 Instantaneous and Average Power

As mentioned before, the instantaneous power p(t) absorbed by an element is the product of the instantaneous voltage v(t) across the element and the instantaneous current i(t) through it. Assuming the passive sign convention,

$$p(t) = v(t)\, i(t) \tag{9.1}$$

The instantaneous power is the power at any instant of time. It is the rate at which an element absorbs energy.

Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as shown in Fig. 9.1. Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v), \qquad i(t) = I_m \cos(\omega t + \theta_i) \tag{9.2}$$

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)\, i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \tag{9.3}$$

We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}\,[\cos(A - B) - \cos(A + B)] \tag{9.4}$$

and express Eq. (9.3) as

$$\mathbf{p(t) = \frac{1}{2}\ VmIm\ cos(\theta v - \theta i) + \frac{1}{2}\ VmIm\ cos(2\omega t + \theta v + \theta i)}$$

$$\mathbf{p(t) = \frac{1}{2}\ VmIm\ cos(\theta v - \theta i) + \frac{1}{2}\ VmIm[cos(2\omega t)\ cos(\theta v + \theta i)\ -\ sin(2\omega t)\ sin(\theta v + \theta i)]} \quad (9.5)$$

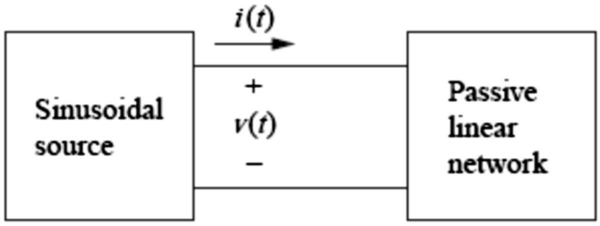

Figure 9.1: Sinusoidal source and passive linear circuit.

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.

A sketch of p(t) in Eq. (9.5) is shown in Fig. 9.2, where T = 2π/ω is the period of voltage or current. We also observe that p(t) is positive for some part of each cycle and negative for the rest of the cycle. When p(t) is positive, power is absorbed by the circuit. When p(t) is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

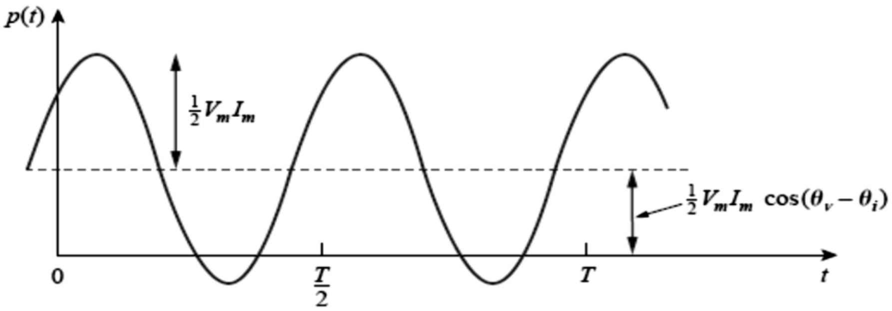

Figure 9.2: The instantaneous power p(t) entering a circuit.

The average power is the average of the instantaneous power over one period.

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\quad dt \quad (9.6)$$

Although Eq. (9.6) shows the averaging done over T, we would get the same result if we performed the integration over the actual period of p(t) which is T0 = T/2.

Substituting p(t) in Eq. (9.5) into Eq. (9.6) gives

$$P = \frac{1}{T}\int_0^T \frac{1}{2} VmIm\cos(\theta v - \theta i)\ dt + \frac{1}{T}\int_0^T \frac{1}{2} VmIm\cos(2\omega t + \theta v + \theta i)\ dt \quad (9.7)$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero. Thus, the second term in Eq. (9.7) vanishes and the average power becomes

$$P = \frac{1}{2} VmIm\cos(\theta v - \theta i) \quad (9.8)$$

Since **cos(θv − θi) = cos(θi − θv)**, what is important is the difference in the phases of the voltage and current.

Example 9.1: Given that v(t) = 120 cos(377t + 45°) V and i(t)= 10 cos(377t − 10°) A find the instantaneous power and the average power absorbed by the passive linear network of Fig. 9.1.

Solution: The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = 1\ 2\ [\cos(A+B) + \cos(A-B)]$$

gives $p(t) = 344.2 + 600\cos(754t + 35^\circ)$ W

The average power is

$$P = \frac{1}{2} VmIm\cos(\theta v - \theta i) = \frac{1}{2}\ 1200\cos[45^\circ - (-10\circ)] = 600\cos 55^\circ = 344.2W$$

Example 9.2: Calculate the average power absorbed by an impedance Z = 30− j 70 Ω when a voltage V = 120 ∠0° is applied across it.

Solution: The current through the impedance is

$$I = \frac{V}{Z} = \frac{120\angle 0^\circ}{30 - j70} = \frac{120\angle 0^\circ}{76.16\angle -66.8^\circ} = 1.576\angle 66.8^\circ\ A$$

The average power is

$$P = \frac{1}{2} VmIm\cos(\theta v - \theta i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24\ W$$

Example 9.3: For the circuit shown in Fig. 9.3, find the average power supplied by the source and the average power absorbed by the resistor

Solution: The current I is given by

$$I = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ\ A$$

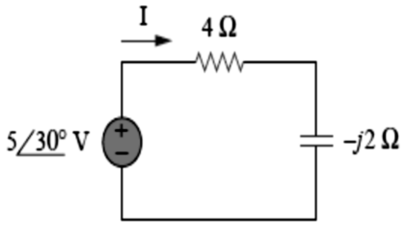

Figure 9.3: For example 9.3.

The average power supplied by the voltage source is

$$\mathbf{P = \frac{1}{2}\ (5)\ (1.118)\ \cos(30^\circ - 56.57^\circ) = 2.5W}$$

The average power absorbed by the resistor is

$$\mathbf{P = I_R \times V_R = \frac{1}{2}\ (4.472)\ (1.118) = 2.5W}$$

Example 9.4: Determine the power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 9.4(a).

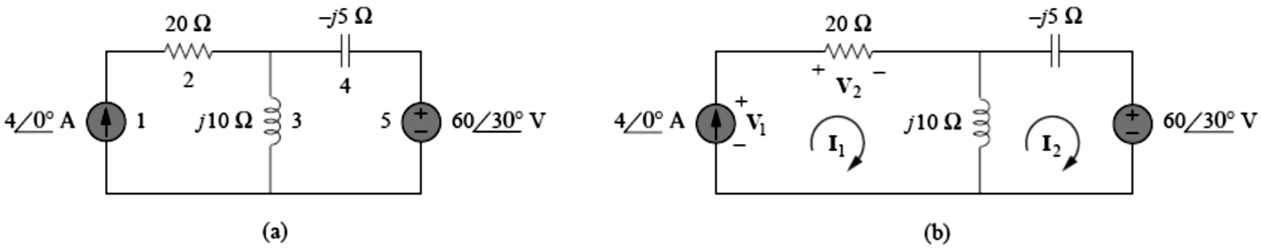

Figure 9.4: For example 9.4.

Solution: We apply mesh analysis as shown in Fig. 9.4(b). For mesh 1,

$$\mathbf{I1 = 4A}$$

For mesh 2,

$$\mathbf{(j10 - j5)\ I2 - j10\ I1 + 60\angle 30^\circ = 0, \qquad I1 = 4A}$$

or $\mathbf{j5\ I2 = -60\angle 30^\circ + j\ 40 \Rightarrow I2 = -12\angle -60^\circ + 8 = 10.58\angle 79.1^\circ\ A}$

For the voltage source, the current flowing from it is **I2 = 10.58 ∠79.1° A** and the voltage across it is **60 ∠30° V**, so that the average power is

$$\mathbf{P5 = \frac{1}{2}\ (60)\ (10.58)\ \cos(30^\circ - 79.1^\circ) = 207.8\ W}$$

the average power is absorbed by the source, in view of the direction of **I2** and the polarity of the voltage source. That is, the circuit is delivering average power to the voltage source.

For the current source, the current through it is **I1 = 4 ∠0°** and the voltage across it is

$$\mathbf{V1 = 20\ I1 + j\ 10(I1 - I2) = 80 + j\ 10(4 - 2 - j10.39) = 183.9 + j20}$$

$$\mathbf{= 184.984\angle 6.21^\circ\ V}$$

The average power supplied by the current source is

$$\mathbf{P1 = -\frac{1}{2}\ (184.984)\ (4)\ \cos(6.21^\circ - 0) = -367.8W}$$

It is negative according to the passive sign convention, meaning that the current source is supplying power to the circuit.

For the resistor, the current through it is I1 = 4 ∠0° and the voltage across it is 20 I1 = 80 ∠0°, so that the power absorbed by the resistor is

$$P2 = \frac{1}{2}\ (80)\ (4) = 160\ W$$

For the capacitor, the current through it is **I2 = 10.58 ∠79.1°** and the voltage across it is **− j5 I2 = (5 ∠−90◦) (10.58 ∠79.1◦) = 52.9 ∠79.1◦ − 90◦.** The average power absorbed by the capacitor is

$$P4 = \frac{1}{2}\ (52.9)\ (10.58)\ \cos(-90^\circ) = 0$$

For the inductor, the current through it is I1 − I2 = 2 − j 10.39 = 10.58 ∠− 79.1◦. The voltage across it is **j10 (I1 − I2) = 105.8 ∠− 79.1◦ + 90◦**. Hence, the average power absorbed by the inductor is

$$P3 = \frac{1}{2}\ (105.8)\ (10.58)\ \cos 90^\circ = 0$$

Notice that the inductor and the capacitor absorb zero average power and that the total power supplied by the current source equals the power absorbed by the resistor and the voltage source, or **P1 + P2 + P3 + P4 + P5 = − 367.8 + 160 + 0 + 0 + 207.8 = 0** indicating that power is conserved.

9.3 Effective or RMS Value

The effectiveness of a voltage or current source in delivering power to a resistive load or R.M.S value is explained in chapter 8. For the sinusoid **i(t) = Im cos ωt**, the effective or **rms** value is

$$Irms = \frac{Im}{\sqrt{2}} \tag{9.9}$$

Similarly, for **v(t)= Vm cos ωt**,

$$Vrms = \frac{Vm}{\sqrt{2}} \tag{9.10}$$

Keep in mind that Eqs. (9.9) and (9.10) are only valid for sinusoidal signals.

The average power in Eq. (9.8) can be written in terms of the rms values.

$$P = \frac{1}{2}\ V\,m\ I\,m\ \cos(\theta v - \theta i) = Vrms\ Irms\ \cos(\theta v - \theta i) \tag{9.11}$$

Example 9.5: Determine the rms value of the current waveform in Fig. 9.5. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.

Solution: The period of the waveform is T = 4. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

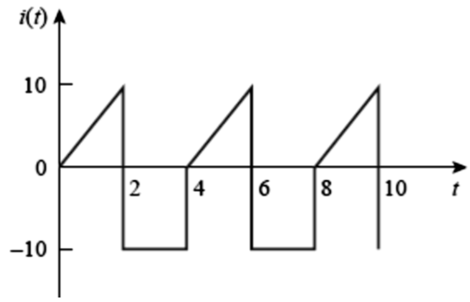

Figure 9.5: For example 9.5.

The **rms** value is

$$\text{Irms} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^T (5t)^2\,dt + \int_0^T (-10)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\Big|_0^2 + 100t\Big|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165\ A$$

The power absorbed by a 2- Ω resistor is

$$P_{av} = I_{rms}^2 \times R = (8.165)^2 \times (2) = 133.3W$$

Example 9.6: The waveform shown in Fig. 9.6 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

Solution: The period of the voltage waveform is T = 2π, and

$$v(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

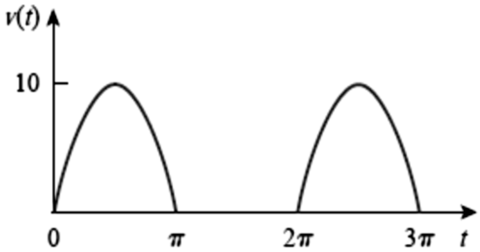

Figure 9.6: For example 9.6.

The rms value is obtained as

$$\text{Vrms} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt}$$

$$= \sqrt{\frac{1}{2\pi}\left[\int_0^\pi (10\sin t)^2\,dt + \int_\pi^{2\pi} (0)^2\,dt\right]}$$

But $\sin^2 t = \frac{1}{2}\,(1 - \cos 2t)$

$$= \sqrt{\frac{1}{2\pi}\left[\int_0^\pi \frac{100}{2}\,(1 - \cos 2t)\,dt\right]} = \sqrt{25} = 5\ V$$

The average power absorbed is

$$P = \frac{V_{rms}^2}{R} = \frac{5^2}{10} = 2.5W$$

9.4 Resistive Circuit

For a purely resistive circuit, v and i are in phase, and $\theta v = \theta i = 0^o$, as appearing in Fig. 9.7. Substituting $\theta v = \theta i = 0^o$ into Eq. (9.5), we obtain

$$\mathbf{P_R = V_{rms}\, I_{rms} \cos(0) + V_{rms}\, I_{rms} \cos(2\omega t + 0)}$$

$$\mathbf{P_R = V_{rms}\, I_{rms} + V_{rms}\, I_{rms} \cos(2\omega t)} \quad (9.12)$$

where $\mathbf{V_{rms}\, I_{rms}}$ is the average or dc term and $\mathbf{V_{rms}\, I_{rms} \cos 2\omega t}$ is a cosine wave with twice the frequency of either input quantity (v or i) and a peak value of $\mathbf{V_{rms}\, I_{rms}}$.

the total power delivered to a resistor will be dissipated in the form of heat.

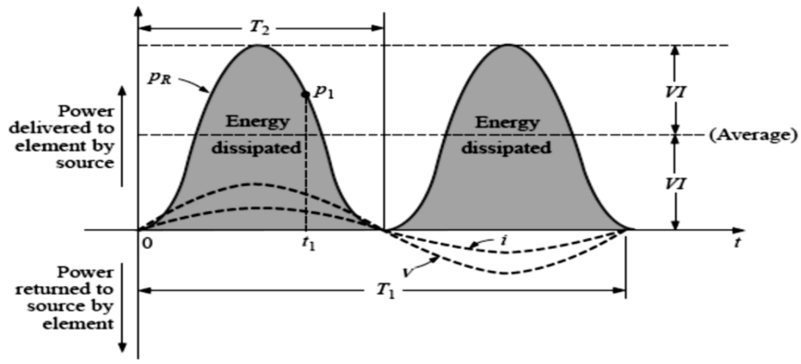

Figure 9.7: Power versus time for a purely resistive load.

The energy dissipated by the resistor (W_R) over one full cycle of the applied voltage (Fig. 9.7) can be found using the following equation:

$$\mathbf{W = Pt}$$

where P is the average value and t is the period of the applied voltage; that is, or, since

$$\mathbf{W_R = V_{rms}\, I_{rms}\, T1} \quad \text{(joules, J)} \quad (9.13)$$

$T1 = 1/f1$,

$$\mathbf{W_R = \frac{V_{rms}\, I_{rms}}{f1}} \quad \text{(joules, J)} \quad (9.14)$$

9.5 Apparent Power and Power Factor

the apparent power is a power rating of significant usefulness in the description and analysis of sinusoidal ac networks and in the maximum rating of a number of electrical components and systems, and is represented symbolically by **S**. Since it is simply the product of voltage and current, its units are volt-amperes, for which the abbreviation is VA.

In Section 9.2 we see that if the voltage and current at the terminals of a circuit are

$$\mathbf{v(t) = Vm \cos(\omega t + \theta v), \text{ and } \quad i(t) = Im \cos(\omega t + \theta i)} \quad (9.15)$$

or, in phasor form, **V = Vm ∠θv** and **I = Im ∠θi**, the average power is

$$\mathbf{P = \frac{1}{2}\ VmIm\ cos(\theta v - \theta i)} \quad (9.16)$$

In Section 9.4, we saw that

$$\mathbf{P = Vrms\ Irms\ cos(\theta v - \theta i) = S\ cos(\theta v - \theta i)} \quad (9.17)$$

We have added a new term to the equation:

$$\mathbf{S = Vrms\ Irms} \quad (9.18)$$

The average power is a product of two terms. The product Vrms Irms is known as the apparent power S. The factor cos(θv − θi) is called the power factor (pf). The apparent power (in VA) is the product of the rms values of voltage and current.

The power factor is dimensionless, since it is the ratio of the average power to the apparent power,

$$pf = \frac{P}{S} = cos(\theta v - \theta i) \quad (9.19)$$

The angle θv − θi is called the power factor angle, since it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance. The impedance is

$$Z = \frac{V}{I} = V\,rms\,I\,rms = Vrms\,Irms\ \angle\theta v - \theta i \quad (9.20)$$

The power factor is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

<u>Example 9.7:</u> A series-connected load draws a current i(t) = 4 cos(100πt + 10°) A when the applied voltage is v(t) = 120 cos(100πt − 20°) V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

<u>Solution:</u> The apparent power is

$$\mathbf{S = Vrms\ Irms = \frac{120}{\sqrt{2}}\,\frac{4}{\sqrt{2}} = 240\ VA}$$

The power factor is

$$\mathbf{pf = cos(\theta v - \theta i) = cos(-20^\circ - 10^\circ) = 0.866\ (leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$Z = \frac{V}{I} = \frac{120\,\angle -20^\circ}{4\,\angle 10^\circ} = 30\angle -30^\circ = 25.98 - j\,15$$

pf = cos(− 30°) = 0.866 (leading)

The load impedance Z can be modeled by a 25.98-Ω resistor in series with a capacitor with

XC = − 15 = − 1 /ωC

or $$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu F$$

Example 9.8: Determine the power factor of the entire circuit of Fig. 9.8 as seen by the source. Calculate the average power delivered by the source.

Solution: The total impedance is

$$Z = 6 + 4\,||(-j\,2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6$$

$$= 7\ \angle -13.24\ \ \Omega$$

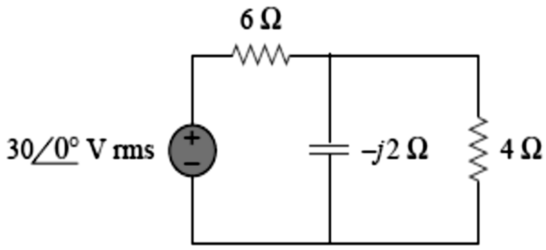

Figure 9.8: For example 9.8.

The power factor is

pf = cos(−13.24) = 0.9734 (leading)

since the impedance is capacitive. The rms value of the current is

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{30\ \angle 0^\circ}{7\ \angle -13.24^\circ} = 4.286\ \angle 13.24^\circ\ A$$

The average power supplied by the source is

P = Vrms Irms pf = (30)(4.286)0.9734 = 125 W

or **$P = I^2_{rms} R = (4.286)^2\ (6.8) = 125$ W**

where R is the resistive part of Z.

9.6 Inductive Circuit and Reactive Power

For a purely inductive circuit, v leads i by 90°, as shown in Fig. 9.9. Therefore, in Eq. (9.5), θi = θv - 90°. Substituting θi = θv - 90° into Eq. (9.5) yields

p_L = VrmsIrms cos(90°) - Vrms Irms[cos(2ωt).cos(90°) - sin(2ωt) sin(90°)]

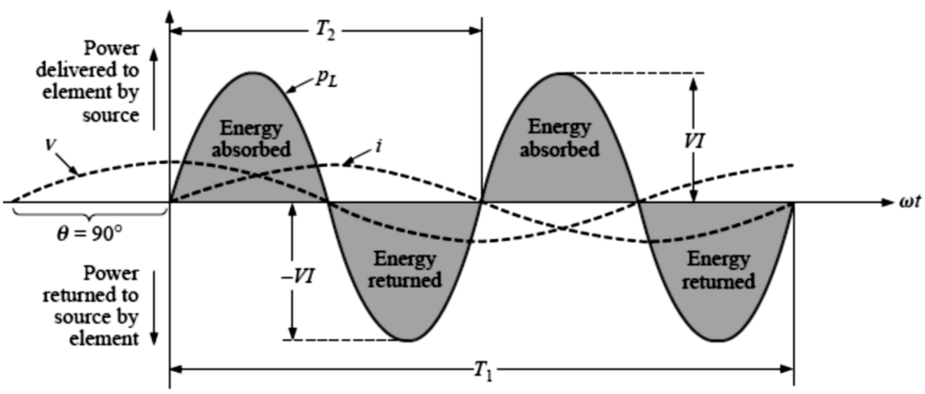

Figure. 9.9: The power curve for a purely inductive load.

or

$$\mathbf{p_L = V_{rms} I_{rms} \sin 2\omega t} \tag{9.21}$$

where **Vrms Irms sin 2ωt** is a sine wave with twice the frequency of either input quantity (v or i). Note the power delivered by the source to the inductor is exactly equal to that returned to the source by the inductor.

The net flow of power to the pure (ideal) inductor is zero over a full cycle, and no energy is lost in the transaction.

The peak value of the curve **Vrms Irms** is defined as the **reactive power** associated with a pure inductor. The symbol for reactive power is Q, and its unit of measure is the volt-ampere reactive (VAR).

$$\mathbf{Q = V_{rms} I_{rms} \sin(\theta v - \theta i)} \quad \text{(volt-ampere reactive, VAR)} \tag{9.22}$$

For the inductor,

$$\mathbf{Q_L = V_{rms} I_{rms}} \quad \text{(VAR)} \tag{9.23}$$

or, since $\mathbf{V = I\, X_L}$ or $\mathbf{I = V/ X_L}$,

$$\mathbf{QL = I^2 X_L} \quad \text{(VAR) or} \quad \mathbf{QL = V^2 / X_L} \quad \text{(VAR)} \tag{9.24}$$

The apparent power associated with an inductor is **S = Vrms Irms**, and the average power is P = 0, as noted in Fig. 9.9. The power factor is therefore

$$\mathbf{pF = \cos(\theta v - \theta i) = P/ S = 0}$$

The energy stored by the inductor during the positive portion of the cycle (Fig. 9.9) is equal to that returned during the negative portion and can be determined using the following equation:

$$\mathbf{W = Pt}$$

where P is the average value for the interval and t is the associated interval of time.

$$\mathbf{W_L = L\, I^2 rms} \tag{9.25}$$

9.7 Capacitive Circuit

For a purely capacitive circuit, i leads v by 90°, as shown in Fig. 9.10. Therefore, in Eq. (9.5), θi = θv + 90°. Substituting θi = θv + 90° into Eq. (9.5), we obtain

$$\mathbf{p_C = -V_{rms} I_{rms} \sin 2\omega t} \tag{9.26}$$

where **– Vrms Irms sin2ωt** is a negative sine wave with twice the frequency of either input (v or i). Again, note the absence of an average or constant term.

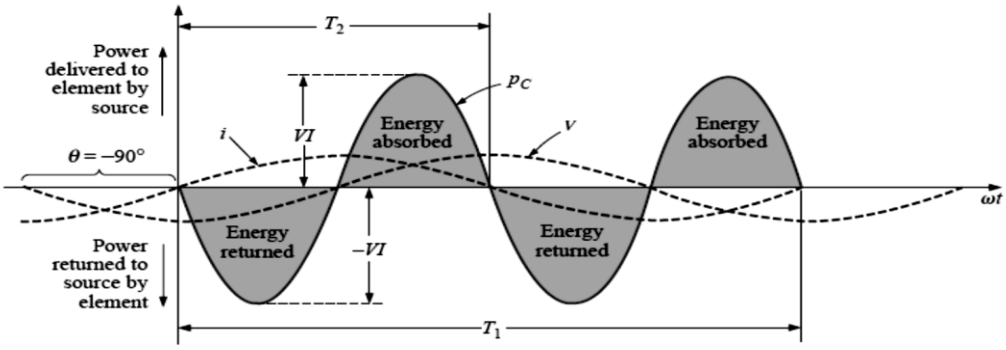

Figure. 9.10: The power curve for a purely capacitive load.

The net flow of power to the pure (ideal) capacitor is zero over a full cycle,

The reactive power associated with the capacitor is equal to the peak value of the p C curve, as follows:

$$Q_C = V_{rms} I_{rms} \quad (VAR) \qquad (9.27)$$

But, since $V = I_{rms} X_C$ and $I = V_{rms}/X_C$, the reactive power to the capacitor can also be written

$$Q_C = I^2_{rms} X_C \quad (VAR), \text{ or } \quad Q_C = V^2_{rms} / X_C \qquad (9.28)$$

and the average power is P = 0, as noted from Eq. (9.26) or Fig. 9.10. The power factor is, therefore,

$$pf = P / S = 0$$

The energy stored by the capacitor during the positive portion of the cycle (Fig. 9.10) is equal to that returned during the negative portion and can be determined using the equation

$$W = Pt.$$

where P is the average value for the interval and t is the associated interval of time

$$W_C = CV^2_{rms}$$

9.8 Maximum Average Power Transfer

In Section 4.6 we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load RL. Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $\mathbf{R_L = R_{Th}}$. We now extend that result to ac circuits. Consider the circuit in Fig. 9.11, where an ac circuit is connected to a load Z_L and is represented by its Thevenin equivalent. The load is usually represented by impedance, which may model an electric motor, an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance Z_{Th} and the load impedance Z_L are

$$\mathbf{Z_{Th} = R_{Th} + jX_{Th}} \qquad (9.29a)$$

$$\mathbf{Z_L = R_L + jX_L} \qquad (9.29b)$$

The current through the load is

$$I = \frac{V_{Th}}{Z_{Th} + Z_L} = \frac{V_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)} \qquad (9.30)$$

$$\mathbf{X_L = -\,X_{Th}} \qquad (9.31)$$

$$\mathbf{R_L = \sqrt{R_{Th}^2 + (X_L^2 + X_{Th}^2)}} \qquad (9.32)$$

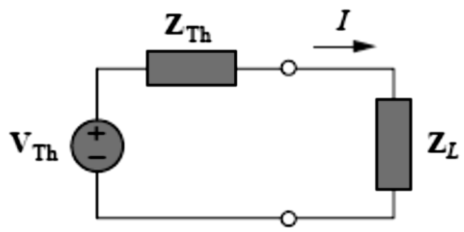

Figure 9.11: Thevenin equivalent to find the maximum average power transfer.

Then, we obtain

$$\mathbf{Z_L = R_L + jX_L = R_{Th} - jX_{Th} = Z^*_{Th}} \qquad (9.33)$$

For maximum average power transfer, the load impedance Z_L must be equal to the complex conjugate of the Thevenin impedance Z_{Th}.

The maximum average power as

$$P_{max} = \frac{|V_{Th}|^2}{8\,R_{Th}} \qquad (9.34)$$

In a situation in which the load is purely real, the condition for maximum power transfer is obtained from Eq. (9.32) by setting $X_L = 0$; that is,

$$\mathbf{R_L = \sqrt{R_{Th}^2 + X_{Th}^2} = |Z_{Th}|} \qquad (9.35)$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

<u>Example 9.9:</u> Determine the load impedance Z_L that maximizes the average power drawn from the circuit of Fig. 9.12. What is the maximum average power?

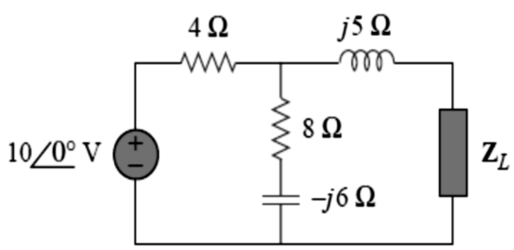

Figure 9.12: For example 9.9.

<u>Solution:</u> First we obtain the Thevenin equivalent at the load terminals. To get Z_{Th}, consider the circuit shown in Fig. 9.13(a). We find

$$\mathbf{Z_{Th} = j5 + 4 \,||\, (8 - j6) = j\,5 + \frac{4(8 - j\,6)}{4 + 8 - j6}}$$

$$\mathbf{= 2.933 + j\,4.467\ \Omega}$$

To find V_{Th}, consider the circuit in Fig. 9.13(b). By voltage division,

$$\mathbf{V_{Th} = \frac{8 - j6}{4 + 8 - j\,6}\,(10) = 7.454\,\angle{-10.3^\circ}\ V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z_L = Z^*_{Th} = 2.933 - j\,4.467\ \Omega}$$

According to Eq. (9.34), the maximum average power is

$$Pmax = \frac{|\,VTh|^2}{8\,RTh} = \frac{(7.454)^2}{8(2.933)} = 2.368\ W$$

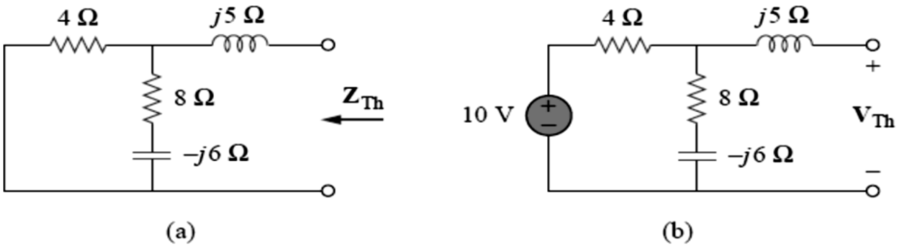

Figure 9.13: Finding the Thevenin equivalent of the circuit in Fig. 9.12.

<u>Example 9.10:</u> In the circuit in Fig. 9.14, find the value of R_L that will absorb the maximum average power. Calculate that power.

Solution: We first find the Thevenin equivalent at the terminals of R_L.

$$ZTh = (40 - j\,30)||j20 = \frac{j20(40 - j\,30)}{j20 + 40 - j\,30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$VTh = \frac{j\,20}{j20 + 40 - j\,30}\,(150\,\angle 30^\circ) = 72.76\,\angle 134^\circ\ V$$

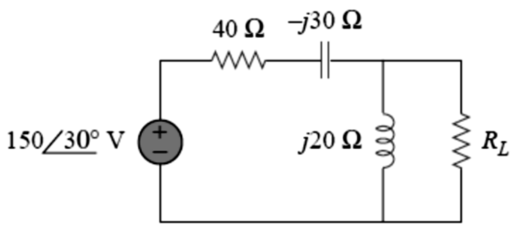

Figure 9.14: For Example 9.10.

The value of $\mathbf{R_L}$ that will absorb the maximum average power is

$$RL = |\,ZTh| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

$$I = \frac{VTh}{ZTh + RL} = \frac{72.76\,\angle 134^\circ}{33.39 + j22.35} = 1.8\,\angle 100.2^\circ\ A$$

The maximum average power absorbed by R_L is

$$\mathbf{Pmax = \frac{1}{2}\,|I|^2\,R_L = \frac{1}{2}\,(1.8)^2\,(24.25) = 39.29\ W}$$

9.9 The Complex Power and Power Triangle

The three quantities **average power, apparent power, and reactive power** can be related in the vector domain by

$$\mathbf{S = P + Q} \qquad (9.36)$$

with $\mathbf{P} = P \angle 0°$ $\mathbf{Q_L} = Q_L \angle 90°$ $\mathbf{Q_C} = Q_C \angle 90°$

For an inductive load, the phasor power **S**, as it is often called, is defined by

$$\mathbf{S} = P + jQ_L$$

as shown in Fig. 9.15. The 90° shift in $\mathbf{Q_L}$ from **P** is the source of another term for reactive power: quadrature power.

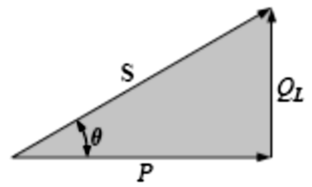

Figure 9.15: Power diagram for inductive loads.

For a capacitive load, the phasor power **S** is defined by

$$\mathbf{S} = P - jQ_C$$

as shown in Fig. 9.15.

If a network has both capacitive and inductive elements, the reactive component of the power triangle will be determined by the difference between the reactive power delivered to each. If $\mathbf{Q_L} > \mathbf{Q_C}$, the resultant power triangle will be similar to Fig. 9.15. If $\mathbf{Q_C} > \mathbf{Q_L}$, the resultant power triangle will be similar to Fig. 9.16.

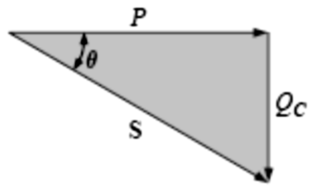

Figure 9.16: Power diagram for capacitive loads.

That the total reactive power is the difference between the reactive powers of the inductive and capacitive elements.

Since the reactive power and average power are always angled 90° to each other, the three powers are related by the Pythagorean theorem; that is,

$$S^2 = P^2 + Q^2 \quad (9.37)$$

Therefore, the third power can always be found if the other two are known.

It is particularly interesting that the equation

$$\mathbf{S} = \mathbf{Vrms}\,\mathbf{I^{*}rms} \quad (9.38)$$

will provide the vector form of the apparent power of a system. Here, **Vrms** is the voltage across the system, and $\mathbf{I^{*}rms}$ is the complex conjugate of the current.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

Complex Power = S = P + jQ = $\frac{1}{2}$VI*

= Vrms Irms ∠ θ v − θ i

Apparent Power = S = |S| = Vrms Irms = $\sqrt{P^2 + Q^2}$

Real Power = P = Re(S) = S cos(θ v − θ i)

Reactive Power = Q = Im(S) = S sin(θ v − θ i)

Power Factor = P/S= cos(θ v − θ i)

<u>Example 9.11:</u> The voltage across a load is **v(t) = 60 cos(ωt − 10◦) V** and the current through the element in the direction of the voltage drop is **i(t) = 1.5 cos(ωt + 50◦) A**. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

<u>Solution:</u>

(a) For the **rms** values of the voltage and current, we write

$$Vrms = \frac{60}{\sqrt{2}} \angle -10^\circ, Irms = \frac{1.5}{\sqrt{2}} \angle +50^\circ$$

The complex power is

$$S = Vrms\, I^{*}rms = \left(\frac{60}{\sqrt{2}} \angle -10^\circ\right)\left(\frac{1.5}{\sqrt{2}} \angle -50^\circ\right) = 45 \angle -60^\circ\, VA$$

The apparent power is

S = |S|= 45 VA

(b) We can express the complex power in rectangular form as

S = 45 ∠− 60° = 45[cos(−60°) + j sin(−60°)] = 22.5 − j 38.97

Since **S** = ***P*** + ***jQ***, the real power is

P = 22.5W

while the reactive power is

Q = − 38.97 VAR

(c) The power factor is

pf = cos(−60°) = 0.5 (leading)

It is leading, because the reactive power is negative. The load impedance is

$$Z = \frac{V}{I} = \frac{60 \angle -10^\circ}{1.5 \angle +50^\circ} = 40 \angle -60^\circ\, \Omega$$

which is a capacitive impedance.

9.10 The Total P, Q , and S

The total number of watts, volt-amperes reactive, and volt-amperes, and the power factor of any system can be found using the following procedure:

1. Find the real power and reactive power for each branch of the circuit.
2. The total real power of the system (P_T) is then the sum of the average power delivered to each branch.
3. The total reactive power (Q_T) is the difference between the reactive power of the inductive loads and that of the capacitive loads.
4. The total apparent power is $S_T = \sqrt{P_T^2 + Q_T^2}$.
5. The total power factor is $\mathbf{P_T/S_T}$.

There are two important points in the above tabulation. **First**, the total apparent power must be determined from the total average and reactive powers and cannot be determined from the apparent powers of each branch, the total apparent power is the sum of complex power of each branch. **Second**, and more important, it is not necessary to consider the series-parallel arrangement of branches. In other words, the total real, reactive, or apparent power is independent of whether the loads are in series, parallel, or series-parallel.

<u>Example 9.12:</u> For the system of Fig. 9.17,

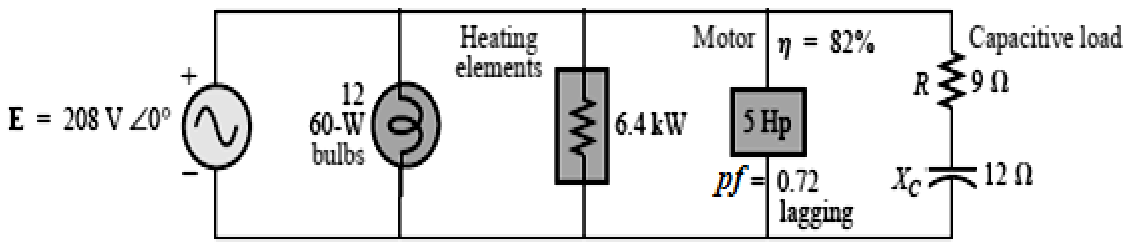

Figure 9.17: For example 9.12.

a. Find the average power, apparent power, reactive power, and **pf** for each branch.

b. Find the total number of watts, volt-amperes reactive, and volt-amperes, and the power factor of the system. Sketch the power triangle.

c. Find the source current I.

<u>Solutions:</u>

a. Bulbs: Total dissipation of applied power

P_1 = 12 (60 W) = 720 W, Q_1 = 0 VAR

$S_1 = P_1$ = 720 VA

pf_1 = 1

Heating elements: Total dissipation of applied power

P_2 = 6.4 kW, Q_2 = 0 VAR

$S_2 = P_2$ = 6.4 kVA

pf_2 = 1

Motor:

$$\eta = \frac{Po}{Pi} \rightarrow Pi = \frac{Po}{\eta} = \frac{5(746W)}{0.82} = 4548.78\ W = P3$$

pf = 0.72 lagging

$$P3 = S3\cos\theta \rightarrow S3 = \frac{P3}{\cos\theta} = 6317.75\ VA$$

Also, **θ = cos^{-1}0.72 = 43.95°**, so that

Q3 = S3 sin θ = (6317.75 VA)(sin 43.95°)

= (6317.75 VA)(0.694) = 4384.71 VAR (L)

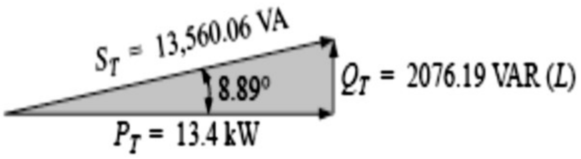

Figure 9.18: Power triangle for example 9.12.

Capacitive load:

$$I = \frac{E}{Z} = \frac{208\ V\angle 0^\circ}{9 - j12} = 13.87\ A\ \angle 53.13^\circ$$

P4 = I^2 R = (13.87 A)^2 . 9 Ω = 1731.39 W

Q4 = I^2 XC = (13.87 A)^2 • 12 Ω = 2308.52 VAR (C)

$$S4 = \sqrt{P_4^2 + Q_4^2} = \sqrt{(1731.39\ W)^2 + (2308.52\ VAR)^2} = 2885.65\ VA$$

pf = 0.6 leading

b. **P_T = P1 + P2 + P3 + P4**

= 720 W + 6400 W +4548.78 W + 1731.39 W = 13,400.17 W

Q_T = ± Q1 ± Q2 ± Q3 ± Q4

= 0 + 0 + 4384.71 VAR (L) − 2308.52 VAR (C) = 2076.19 VAR (L)

$$ST = \sqrt{P_T^2 + Q_T^2} = \sqrt{(13,400.17\ W)^2 + (2076.19\ VAR)^2} = 13,560.06\ VA$$

pf_T = 0.988 lagging

θ = cos^{-1} 0.988 = 8.89°

Note Fig. 9.18.

c. $ST = E\,I \rightarrow I = \frac{ST}{E} = 65.19\,A$

Lagging power factor: E leads I by 8.89°, and

I = 65.19 A ∠- 8.89 A

Example 9.13: Figure 9.19 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the (4+ j2) impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

Solution: The total impedance is

$$Z = (4 + j2) + (15 - j10) = 19 - j\,8 = 20.62\,\angle -22.83^\circ\ \Omega$$

The current through the circuit is

$$I = \frac{Vs}{Z} = \frac{220\,\angle 0^\circ}{20.62\,\angle -22.83^\circ}$$

$$= 10.67\,\angle 22.83^\circ\ A\,rms$$

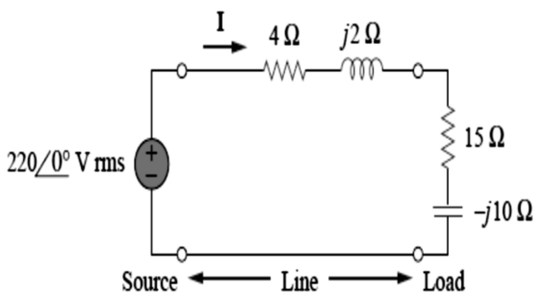

Figure 9.19: For example 9.13.

(a) For the source, the complex power is

$$S_s = V_s\, I^* = (220\,\angle 0^\circ)(10.67\,\angle -22.83^\circ)$$

$$= 2347.4\,\angle -22.83^\circ = (2163.5 - j910.8)\ VA$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$V_{line} = (4 + j2)I = (4.472\,\angle 26.57^\circ)(10.67\,\angle 22.83^\circ) = 47.72\,\angle 49.4^\circ\ V\ rms$$

The complex power absorbed by the line is

$$S_{line} = V_{line}\, I^* = (47.72\,\angle 49.4^\circ)(10.67\,\angle -22.83^\circ)$$

$$= 509.2\,\angle 26.57^\circ = 455.4 + j\,227.7\ VA$$

or $S_{line} = |I|^2\, Z\ line = (10.67)^2\,(4 + j2) = 455.4 + j\,227.7\ VA$

That is, the real power is 455.4 W and the reactive power is 227.76 VAR (lagging).

(c) For the load, the voltage is

$$V_L = (15 - j10)I = (18.03\,\angle -33.7^\circ)(10.67\,\angle 22.83^\circ)$$

$$= 192.38\,\angle -10.87^\circ\ V\ rms$$

The complex power absorbed by the load is

$$S_L = VLI* = (192.38\,\angle -10.87^\circ)(10.67\,\angle -22.83^\circ)$$

$$= 2053\,\angle -33.7^\circ = (1708 - j\,1139)\ VA$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S_s = S_{line} + S_L}$, as expected. We have used the rms values of voltages and currents.

9.11 Power Factor Correction

Most domestic loads (such as washing machines, air conditioners, and refrigerators) and industrial loads (such as induction motors) are inductive and operate at a low lagging power factor. Although the inductive nature of the load cannot be changed, we can increase its power factor.

Since most loads are inductive, as shown in Fig. 9.20(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. 9.20(b). The effect of adding the capacitor can be illustrated using either the power triangle or the phasor diagram of the currents involved. Figure 9.21 shows the latter, where it is assumed that the circuit in Fig. 9.20(a) has a power factor of cos θ1, while the one in Fig. 9.20(b) has a power factor of cos θ2. It is evident from Fig. 9.21 that adding the capacitor has caused the phase angle between the supplied voltage and current to reduce from θ1 to θ2, thereby increasing the power factor.

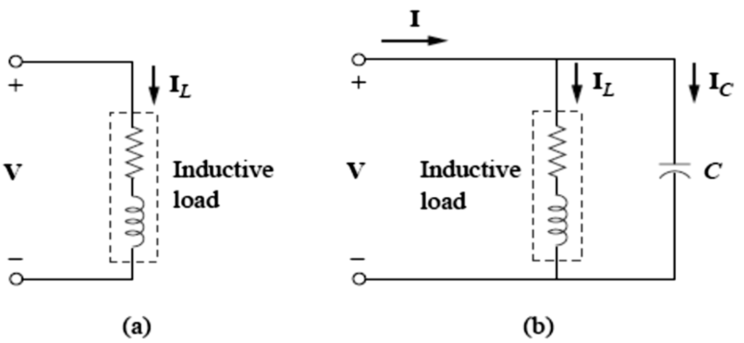

Figure 9.20: Power factor correction: (a) original inductive load, (b) inductive load with improved power factor.

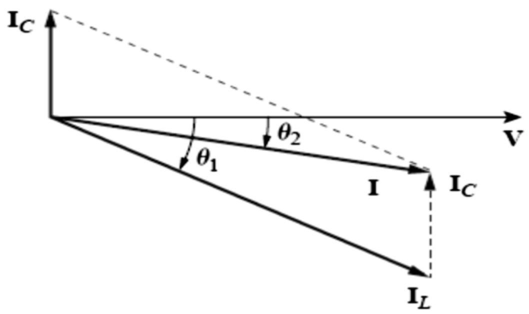

Figure 9.21: Phasor diagram showing the effect of adding a capacitor in parallel with the inductive load.

We can look at the power factor correction from another perspective. Consider the power triangle in Fig. 9.22. If the original inductive load has apparent power S1, then

$$\mathbf{P = S_1 \cos\theta_1, \qquad Q_1 = S_1 \sin\theta_1 = P \tan\theta_1} \tag{9.39}$$

If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2 \cos\theta_2$), then the new reactive power is

$$\mathbf{Q_2 = P \tan\theta_2} \tag{9.40}$$

The reduction in the reactive power is caused by the shunt capacitor, that is,

$$Q_C = Q_1 - Q_2 = P(\tan \theta_1 - \tan \theta_2) \quad (9.41)$$

But, $Q_C = V^2rms / X_C = \omega C V^2 rms$. The value of the required shunt capacitance C is determined as

$$C = \frac{QC}{\omega V^2 rms} = \frac{P(\tan \theta 1 - \tan \theta 2)}{\omega V^2 rms} \quad (9.42)$$

Note that the real power P dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

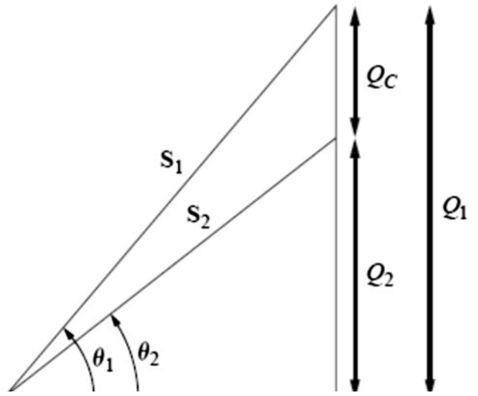

Figure 9.22: Power triangle illustrating power factor correction.

Example 9.14: When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Solution:

If the pf = 0.8, then

$$\cos \theta_1 = 0.8 \Rightarrow \theta_1 = 36.87^\circ$$

where θ_1 is the phase difference between voltage and current. We obtain the apparent power from the real power and the **pf** as

$$S_1 = \frac{P}{\cos \theta_1} = \frac{4000}{0.8} = 5000\, VA$$

The reactive power is

$$Q_1 = S_1 \sin \theta_1 = 5000 \sin 36.87 = 3000 \text{ VAR}$$

When the pf is raised to 0.95,

$$\cos \theta 2 = 0.95 \Rightarrow \theta 2 = 18.19^\circ$$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos \theta_2} = \frac{4000}{0.95} = 4210.5\, VA$$

The new reactive power is

$$Q_2 = S_2 \sin \theta_2 = 1314.4 \text{V AR}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{V A R}$$

and $C = \frac{QC}{\omega V^2 rms} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu F$