Laplace transform:

is a mathmatical technique that changes a function of time into a function in the frequency domain.

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} . f(t)\, dt$$

Some properties of laplace transform:

(1) $L\{C_1 f_1(t) + C_2 f_2(t)\} = C_1 L\{f_1(t)\} + C_2 L\{f_2(t)\}$
$= C_1 F_1(s) + C_2 F_2(s)$

(2) $L\{e^{at} . f(t)\} = F(s-a)$

(3) $L\left\{\int_0^t f(u)\, du\right\} = \dfrac{F(s)}{s}$

(4) $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^{\infty} f(u)\, du$

(5) $L\{f'(t)\} = s\, f(s) - f(0)$

$L\{f''(t)\} = s^2 f(s) - s\, f(0) - f'(0)$

$L\{f^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots$

(6) If $L\{f(t)\} = F(s)$ then

$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

ex. 1/ $L\{4t^5 - 2t + 6t^3\}$

sol/

$$F(s) = 4\,\frac{5!}{S^{5+1}} - 2\,\frac{1}{S^{1+1}} + 6\,\frac{3!}{S^{3+1}}$$

$$F(s) = \frac{480}{S^6} - \frac{2}{S^2} + \frac{36}{S^4}$$

EX. 2/ $L\{e^{2t}.\, t^2\}$

$$F(s) = L\{t^2\} = \frac{2!}{S^{2+1}} = \frac{2}{S^3}$$

$$L\{e^{2t}.\, t^2\} = \frac{2}{(S-2)^3}$$

EX. 3/ $L\{e^{-3t}.\, \sin 2t\}$

$$L\{\sin 2t\} = F(s) = \frac{2}{S^2+4}$$

$$L\{e^{-3t}.\, \sin 2t\} = \frac{2}{(S+3)^2+4} = \frac{2}{S^2+6S+13}$$

EX. 4/ $L\left\{\int_0^t \sin 2u\, du\right\}$

$$L\{\sin 2u\} = \frac{2}{S^2+4}$$

$$L\left\{\int_0^t \sin 2u\, du\right\} = \frac{2}{S(S^2+4)}$$

EX.5/ $L\left\{\frac{\sin t}{t}\right\}$

$L\{\sin t\} = \frac{1}{s^2+1}$

$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{1}{u^2+1}\, du$

$= \left[\tan^{-1} u\right]_s^{\infty} = \tan^{-1}\infty - \tan^{-1} s$

$= \frac{\pi}{2} - \tan^{-1} s$

EX.6/ Find $L\{(\sin t - \cos t)^2\}$

$L\{\sin^2 t - 2\sin t \cos t + \cos^2 t\} = L\{1 - \sin 2t\}$

$= \frac{1}{s} + \frac{2}{s^2+4}$

EX.7/ If $L\{f(t)\} = \frac{s^2 - s + 1}{(2s+1)^2 (s-1)}$, find $L\{f(2t)\}$?

Sol:-

$L\{f(2t)\} = \frac{1}{2} f\left(\frac{s}{2}\right)$

$= \frac{1}{2} \cdot \frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{(s+1)^2 \left(\frac{s}{2} - 1\right)}$

EX. 8/ Find $L\left\{e^{-3t}\int_0^t \frac{\sin 2t}{t}\,dt\right\}$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty \frac{2}{u^2+4}\,du = \tan^{-1}\frac{u}{2}\Big]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$L\left\{\int_0^t \frac{\sin 2t}{t}\,dt\right\} = \frac{\pi}{2s} - \frac{1}{s}\tan^{-1}\frac{s}{2}$$

$$L\left\{e^{-3t}\int_0^t \frac{\sin 2t}{t}\,dt\right\} = \frac{\pi}{2(s+3)} - \frac{1}{s+3}\tan^{-1}\frac{s+3}{2}$$

H.W 3// Find laplace transform of the following:

(1) $\frac{e^{3t}\cos 4t}{t}$

(2) $5e^{2t} - t^3$

(3) $t^2 - 7 + \cosh 3t$

Unit step function:

Theory: $L[f(t)\ u(t-a)] = e^{-as} L[f(t+a)]$

where: $u(t-a)$ is unit step function.

EX.(1) Find $L\{\cos(t-1)\ u(t+1)\}$

sol:-

$f(t) = \cos(t-1)$ where $a=1$

$f(t+1) = \cos(t+1-1) = \cos t$

$\therefore L\{\cos(t-1)\ u(t-1)\} = e^{-s} L\{\cos t\}$

$= e^{-s} \dfrac{s}{s^2+1}$

EX.(2) Find $L\{t^2\ u(t-2)\}$

$f(t) = t^2$ where $a=2$

$f(t+2) = (t+2)^2$

$= t^2+4t+4$

$L\{t^2\ u(t-2) = e^{-2s}\left\{\dfrac{2!}{s^3} + \dfrac{4}{s^2} + \dfrac{4}{s}\right\}$

EX.(3) Find $L\{(t^2-1)\ u(t-1)\}$

sol:

$f(t) = t^2 - 1$ $a=1$

$f(t+1) = (t+1)^2 - 1$

$= t^2 + 2t$

$$L\{(t^2-1)\ u(t-1)\} = e^{-s}\left[\frac{2!}{s^3} + \frac{2}{s^2}\right]$$

EX.(4) Find $L\{(t^2-4t+5)\ u(t-3)\}$

sol:

$f(t) = t^2 - 4t + 5$ $a=3$

$f(t+3) = (t+3)^2 - 4(t+3) + 5$

$$= e^{-3s} L\{t^2 + 6t + 9 - 4t - 12 + 5\}$$

$$= e^{-3s} L\{t^2 + 2t + 2\}$$

$$= e^{-3s}\left\{\frac{2!}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right\}$$