Al-Mustaqbal university

Engineering technical college

Department of Building &Construction Engineering

Mathematics

First class

Lecture No.13

Define integral

Assist. Lecture

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Introduction:

In mathematics, a matrix is also known as matrices. It is a rectangular array of numbers, figures, or expressions, organized in rows and columns. Matrices are usually written in box brackets. In matrices, the horizontal and vertical lines of entries are rows and columns. The size of a matrix is determined by the number of rows and columns that it holds. A matrix with m rows and n columns is named an m × n matrix or M-by-N matrix, while m and n are described its dimensions. The dimensions of the resulting matrix are 2 × 3 up (read "two by three") as there are 2 rows and 3 columns.

the individual parts that are the numbers, symbols, or expressions in a matrix are named as their entries.

Definition:

- A matrix is a set of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrix algebra has at least two advantages:

- Reduces complicated systems of equations to simple expressions
- Adaptable to systematic method of mathematical treatment and well suited to computers

Properties:

1. Location: the first index represent row C_i and the second index represent the column C_j

   e.g.($a_{21} is\ the\ element\ in\ row\ 2.column\ 1\ of\ the\ matrix\ A$)

2. Value: which represent the amount of the element.

- **Examples:**

- 3x3 matrix $\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix}$

- 2x4 matrix $\begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix}$
- 1x2 matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}$

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters.

A general real matrix $A \in R$ with $m \times n$ elements is of the form:

$$A_{mxn} = \begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{in} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m   j goes from 1 to n

Type of Matrices:

1. Column matrix:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

2. Row matrix:

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13}\cdots & a_{1n} \end{bmatrix}$$

3. Rectangular matrix:

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix} \qquad m \neq n$$

4. **Square matrix:**

The number of rows is equal to the number of columns

(a square matrix A has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

5. **Unit or Identity matrix - I**

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. a_{ij} =0 for all $i = j$

a_{ij} = 1 for some or all $i = j$

6. **Null (zero) matrix - 0**

All elements in the matrix are zero

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_{ij} = 0$ For all i,j

7. **Triangular matrix**

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

7a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

7b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

8. Diagonal matrix

Only has non-zero elements on the main diagonal, those non-zero elements can have any value.

$$\begin{bmatrix} d_{ij} & 0 & 0 \\ 0 & d_{ij} & 0 \\ 0 & 0 & d_{ij} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i = j$ $\qquad$ $a_{ij} = a$ for all $i = j$

Matrices – Operations

1. Equality of Matrices

Two matrices are said to be equal only when all corresponding elements are equal

Therefore, their size or dimensions are equal as well

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad A=B$$

Some properties of equality:

- If A = B, then B = A for all A and B
- If A = B, and B = C, then A = C for all A, B and C
- A= $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ B= $\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$
- If A = B then $a_{ij} = b_{ij}$

2. <u>Addition and Subtraction of Matrices</u>

- The sum or difference of two matrices, A and B of the same size yields a matrix C of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Commutative Law:

A + B = B + A

Associative Law:

A + (B + C) = (A + B) + C = A + B + C

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A B C

2x3 2x3 2x3

A + 0 = 0 + A = A

A + (-A) = 0 (where –A is the matrix composed of –a_{ij} as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

3. Scalar Multiplication of Matrices

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then kA = Ak

Example: If k=4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

$$4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$$

Properties:

k (A + B) = kA + kB

(k + g)A = kA + gA

k(AB) = (kA)B = A(k)B

4. Multiplication of Matrices

The product of two matrices is another matrix

Two matrices A and B must be conformable for multiplication to be possible

i.e. the number of columns of A must equal the number of rows of B

Example 1:

A x B = C

(1x3) (3x1) (1x1)

A x B = Not possible!

(6x2) (6x3)

Example 2:

A x B = C

(2x3) (3x2) (2x2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of A with column *j* of B – row by column multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times4)+(2\times6)+(3\times5) & (1\times8)+(2\times2)+(3\times3) \\ (4\times4)+(2\times6)+(7\times5) & (4\times8)+(2\times2)+(7\times3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Assuming that matrices A, B and C are conformable for the operations indicated, the following are true:

1. A(BC) = (AB)C = ABC - (associative law)
2. A(B+C) = AB + AC - (first distributive law)
3. (A+B)C = AC + BC - (second distributive law)

- **Caution!**
- **AB** not generally equal to **BA**, **BA** may not be conformable
- If **AB** = **0**, neither **A** nor **B** necessarily = **0**
- If **AB** = **AC**, **B** not necessarily = **C**

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

- If AB = 0, neither A nor B necessarily = 0

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **<u>Transpose of A Matrix</u>**

If:

$$A = {}_{2}A^{3} = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

2x3

Then transpose of A, denoted A^T is:

$$A^T = {}_{2}A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$a_{ij} = a^T_{ji}$ For all i and j

To transpose:

Interchange rows and columns

The dimensions of A^T are the reverse of the dimensions of A

$$A = {}_{2}A^{3} = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$ 2 x 3

$$A^T = {}_3A^{T^{2}} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix} \qquad 3 \text{ x } 2$$

- **<u>Properties of transposed matrices:</u>**

1. $(A+B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(A^T)^T = A$

   1. $(A+B)^T = A^T + B^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

   2. $(AB)^T = B^T A^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

- **<u>Determinant of A Matrix</u>**

1. The determine of a 2 x 2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is written det.

   A or $|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

   − +

$= a_{11} * a_{22} - a_{12} * a_{21}$

Example 1:

Find the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & -7 \end{bmatrix}$

|A|= (1*-7) - (2*4) = -15

2. The determine of a 3 x 3 matrix is written as:

$$|\mathrm{A}| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$|\mathrm{A}| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 2:

Find det. A if $\mathrm{A} = \begin{bmatrix} 3 & -5 & 3 \\ 2 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix}$

Solution:

$$|\mathrm{A}| = 3 \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} - (-5) \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} + 3 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|\mathrm{A}| = 3\big(1 * 4 - (-1 * 0)\big) - (-5)\big(2 * 4 - (-1 * 1)\big) + 3\big(2 * 0 - (1 * 1)\big)$$

$$12 + 45 - 3 = 54$$

There is another to compute the determinant of a 3 x 3 matrix. It is named Sarrus' rule or Sarrus' Scheme.

Consider a 3 x 3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then it is det. Can be computed by the following Scheme:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

- - - + + +

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{22}a_{32} - a_{12}a_{21}a_{33}$$

Example 3:

$$|\mathrm{A}| = \begin{bmatrix} 3 & -5 & 3 \\ 2 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix}\begin{matrix} 3 & -5 \\ 2 & 1 \\ 1 & 0 \end{matrix}$$

$$= 3 * 1 * 4 + (-5 * -1 * 1) + 3 * 2 * 0 - (3 * 1 * 1) - (3 * -1 * 0) - (-5 * 2 * 4) = 54$$

- **Cofactors**

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when $i+j$ is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1,\ j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1,\ j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1,\ j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

- **Adjoint Matrices**

A cofactor matrix C of a matrix A is the square matrix of the same order as A in which each element a_{ij} is replaced by its cofactor c_{ij}.

$$(adj\ A) = [\![Cof(A)]\!]^T$$

Example:

Compute the $adjA$ given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

$$C_{11} = (-\mathbf{1})^{\mathbf{2}} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$$

$$C_{12} = (-\mathbf{1})^{\mathbf{3}} \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$C_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2$$

$$C_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

$$A = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$adjA = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$