عنوان المحاضرة:

Integration, integrals of trigonometric & inverse functions, Integrals of logarithm & Exponential Functions

التكامل، تكامل الدوال المثلثية ومعكوسها، تكامل الدوال اللوغارتمية والاسية

Integration :- التكامل

The process of integration reverses the process of differentiation.

If $F(x) = 2x^2 \longrightarrow f'(x) = 4x$.
The integration of $4x$ is $2x^2$.

Integration is a process of summation or adding parts together & an elongated S, shown as $\int$, is used to replace the words "the integral of"

Types of Integration :- انواع التكامل

1. **Indefinite Integrals** التكامل غير محدد
   Integrals containing an arbitrary constant "C" in their results. This constant needs further infos to be found/calculated.
2. **Definite Integrals** التكامل محدد
   Integration limits are applied (يتم تطبيق حدود التكامل)

If an expression is written as $[x]_a^b$, b is called the upper limit and a is the lower limit, where,

$$[x]_a^b = b - a$$

The process of Integration

In integration, the variable of Integration is shown by adding d (the derivative) after the Function to be integrated.

Thus, $\int 4x\,dx$ means "the integral of $4x$ with respect to x".

and $\int 2t\,dt$ means "the integral of $2t$ with respect to t"

So,

$$\int dx = x + C$$
$$\int dy = y + C$$
$$\int dt = t + C$$

Standard Integrals

① Integral of constant $\Longrightarrow \int a\,dx = ax + C$, a-cons

② Power raised variable $\Longrightarrow \int ax^n dx = \frac{ax^{n+1}}{n+1} + C$

Examples

① $\int 3x^2 dx \Longrightarrow \int 3x^2 dx = \frac{3x^{2+1}}{2+1} + C = \frac{3}{3}x^3 + C$

$= \boxed{x^3 + C}$

② $\int 3x^4 dx = \frac{3x^{4+1}}{4+1} + C = \boxed{\frac{3}{5}x^5 + C}$

③ $\int \frac{2}{x^2} dx = \int 2x^{-2} dx = \frac{2x^{-2+1}}{-2+1} + C = \frac{2x^{-1}}{-1} + C$

$= \boxed{\frac{-2}{x} + C}$

④ $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{3/2}}{3/2} + C$

$= \boxed{\frac{2}{3}\sqrt{x^3} + C}$

⑤ $\int (3x + 2x^2 - 5)\,dx = \int 3x\,dx + \int 2x^2\,dx - \int 5\,dx$

$= 3\frac{x^{1+1}}{1+1} + 2\frac{x^{2+1}}{2+1} - 5x + C$

$= \boxed{\frac{3x^2}{2} + \frac{2x^3}{3} - 5x + C}$

Integrals of the Trigonometric functions :- تكاملات الدوال المثلثية

① $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

② $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

③ $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

④ $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

⑤ $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$

⑥ $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$

Examples

① $\int [8\cos x + 3\sin x]\,dx$

Sol

$= 8\sin x + 3(-\cos x) + C$

$= \boxed{8\sin x - 3\cos x + 1}$

② $\int [4\sec^2 x - \sec x \tan x]\,dx$

Sol

$= \boxed{4\tan x - \sec x + C}$

③ $\int \csc x(\cot x - \csc x)\,dx = \int (\csc x \cot x - \csc^2 x)\,dx$

$= \csc x - (-\cot x) + C = \boxed{\cot x - \csc x + C}$

④ $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$

$= \int (1-\sin^2 x)\cos x\,dx$

$= \int \cos x\,dx - \int \sin^2 x \cos x\,dx$

$= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$

⑤ $\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx$

$= \int (\cos^2 x)^2 \cos x\,dx$

$= \int (1-\sin^2 x)^2 \cos x\,dx$

let $u = \sin x$

$du = \cos x\,dx$

$\therefore = \int (1-u^2)^2\,du = \int (1-u^2)(1-u^2)\,du$

$= \int (1-2u^2+u^4)\,du$

$= u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$

replace u with $\sin x$ & du with $\cos x\,dx$

$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C}$

Integration of Inverse Trigonometric Functions

① $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$

② $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

③ $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$

④ $\int \frac{-du}{\sqrt{a^2-u^2}} = \cos^{-1}\left(\frac{u}{a}\right) + C$

⑤ $\int \frac{-du}{a^2+u^2} = \frac{1}{a}\cot^{-1}\left(\frac{u}{a}\right) + C$

⑥ $\int \frac{-du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\csc^{-1}\left(\frac{u}{a}\right) + C$

Examples

① $\int \frac{dx}{\sqrt{16-x^2}}$ Sol. $\Longrightarrow$ $a^2=16 \Longrightarrow a=4$

$u=x$

$\therefore \int \frac{dx}{\sqrt{16-x^2}} = \boxed{\sin^{-1}\frac{x}{4}+c}$

② $\int \frac{3}{25+x^2}dx$ Sol. $\Longrightarrow$ $a^2=25 \Longrightarrow a=5$

$u=x$

$\therefore \int \frac{3\,dx}{25+x^2} = 3\int \frac{dx}{25+x^2} = 3\times\frac{1}{5}\tan^{-1}\frac{x}{5}+c$

$= \boxed{\frac{3}{5}\tan^{-1}\frac{x}{5}+c}$

③ $\int \frac{8}{x\sqrt{4x^2-1}}dx$ Sol. $\Longrightarrow$ $a^2=1 \Longrightarrow \boxed{a=1}$

$u^2=4x^2 \Longrightarrow \boxed{u=2x}$

$\therefore \int \frac{8}{x\sqrt{4x^2-1}}dx = 8\int \frac{2dx}{2x\sqrt{4x^2-1}} = \boxed{8\sec^{-1}(2x)+c}$

Another method $du=2dx \Longrightarrow dx=\frac{du}{2}$, $\boxed{x=\frac{u}{2}}$

$8\int \frac{dx}{x\sqrt{4x^2-1}} = 8\int \frac{du/2}{\frac{u}{2}\sqrt{u^2-1}} = 8\int \frac{du}{u\sqrt{u^2-1}}$

$= 8\sec^{-1}u+c$

$= \boxed{8\sec^{-1}(2x)+c}$

Standard integration of inverse Tri Fcns so

① $\int \sin^{-1}x\,dx = x\sin^{-1}x+\sqrt{1-x^2}+c$

② $\int \cos^{-1}x\,dx = x\cos^{-1}x-\sqrt{1-x^2}+c$

③ $\int \tan^{-1}x\,dx = x\tan^{-1}x-\frac{1}{2}\ln|1+x^2|+c$

④ $\int \sec^{-1}x\,dx = x\sec^{-1}x-\ln|x+\sqrt{x^2-1}|+c$

⑤ $\int \csc^{-1}x\,dx = x\csc^{-1}x+\ln|x+\sqrt{x^2-1}|+c$

⑥ $\int \cot^{-1}x\,dx = x\cot^{-1}x+\frac{1}{2}\ln|1+x^2|+c$

Integration of Logarithm & Exponential Functions

تكامل الدوال اللوغارتمية والاسية

① $\int \log_a u \, du = \frac{u \cdot \log_a(\frac{u}{e})}{u'} + C$ (u = linear Function)

Ex① $\int \log_4 x \, dx$

Sol.)

$u = x \longrightarrow u' = 1$

$a = 4$

$\therefore \int \log_4 x \, dx = \frac{x \log_4(\frac{x}{e})}{1} + c = \boxed{x \log_4(\frac{x}{e}) + c}$

Ex② $\int \log_5(x+7) \, dx$

Sol.)

$u = x+7 \longrightarrow u' = 1$

$a = 5$

$\therefore \int \log_5(x+7) \, dx = \frac{(x+7) \log_5(\frac{x+7}{e})}{1} + c = \boxed{(x+7) \log_5(\frac{x+7}{e}) + c}$

Ex② $\int \log_7 x^4 \, dx$

Sol.)

In such a problem, we need to manipulate the log before going ahead with using the formula ①

$\int \log_7 x^4 dx = 4 \int \log_7 x \, dx \Rightarrow$ ($u = x \rightarrow u' = 1$, $a = 7$)

$= \frac{4 \cdot x \log_7(\frac{x}{e})}{1} + c$

$= \boxed{4x \log_7(\frac{x}{e}) + c}$

(Ex) $\int \log_2(x^2+8x+16)\,dx$

Sol:-

$$\int \log_2(x^2+8x+16)\,dx = \int \log_2(x+4)^2\,dx$$

$$= 2\int \log_2(x+4)\,dx \qquad , \quad u = x+4 \Rightarrow u' = 1, \; a = 2$$

$$= \boxed{2(x+4)\log_2\left(\frac{x+4}{e}\right) + C}$$

---

(2) $\int \frac{1}{x}\,dx = \ln|x| + C$

(3) $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

Examples

(1) $\int \frac{7}{x}\,dx = 7\int \frac{1}{x}\,dx = \boxed{7\ln x + C}$

(2) $\int \frac{1}{x+5}\,dx = \boxed{\ln(x+5) + C}$

(3) $\int \frac{5}{6-2x}\,dx = \int \frac{5}{6-2x} * \frac{-2}{-2} = \frac{-1}{2}\int \frac{5(-2)}{6-2x}\,dx$

$$= \boxed{\frac{-5}{2}\ln(6-2x) + C}$$

(4) $\int \frac{x}{x^2-3}\,dx = \int \frac{x\,dx}{x^2-3} * \frac{2}{2} = \frac{1}{2}\int \frac{2x}{x^2-3}\,dx$

$$= \boxed{\frac{1}{2}\ln(x^2-3) + C}$$

(5) $\int e^{2x}\,dx = \int e^{2x}\,dx * \frac{2}{2} = \frac{1}{2}\int e^{2x} * 2\,dx$

$$= \boxed{\frac{1}{2}e^{2x} + C}$$

(6) $\int e^{-5x}\,dx = \boxed{\frac{-1}{5}e^{-5x} + C}$

نهاية محاضرة " Integration, integrals of trigonometric & inverse Exponential Functions & functions, Integrals of logarithm التكامل، تكامل الدوال المثلثية ومعكوسها، تكامل الدوال اللوغارتمية والاسية"