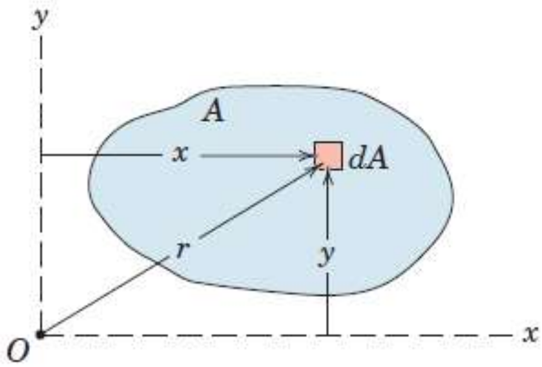
# Chapter Seven: Second Moment or Moment of Inertia

## 7.1 Definitions

Moment of inertia, also called the second moment of the element of area $dA$ with respect to any axis, is the product of area of element and the square of distance from the axis to the element. The moments of inertia of the element $dA$ about the $x-$ and $y-$ axes are, by definition, $\boldsymbol{dI_x = y^2\, dA}$ and $\boldsymbol{dI_y = x^2\, dA}$, respectively.




The moments of inertia of area (A) about the same axes are:

$I_x = \int y^2\ dA$ Moment of inertia about the $x-$ axis

$I_y = \int x^2\ dA$ Moment of inertia about the $y-$ axis

The expressions $I_x,\ I_y$ are called rectangular moments of inertia. if the moment axis is perpendicular to the plane of the area, the second moment of area is called polar moment of inertia, that is:

$$dI_o = r^2\, dA$$

$$I_o = \int (x^2 + y^2) dA = \int x^2\ dA + \int y^2\ dA$$

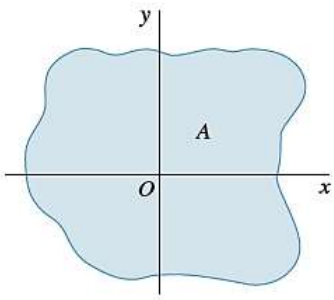
$I_o \ = I_y + I_x$ polar moment of inertia

where $I_o$ is the polar moment of inertia of area with respect to an axis through point (o) perpendicular to the plane of area.

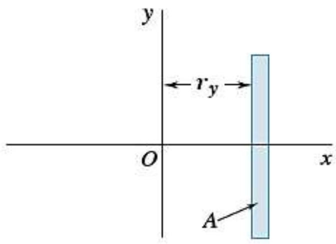
The second moment of area has dimensions of length raised to the forth power ($L^4$). Also, the second moment of an area is always a positive quantity.

## 7.2 Radius of Gyration

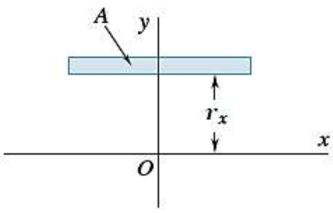
The radius of gyration of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. Provided the areas and moments of inertia are known, the radii of gyration are determined from the formulas:



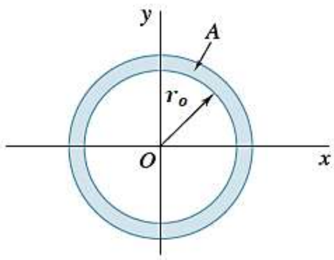

(a) Area A with given moment of inertia




(c) Compressing the area to a vertical strip with radius of gyration $r_y$




(b) Compressing the area to a horizontal strip with radius of gyration $r_x$;




(d) Compressing the area to a circular ring with polar radius of gyration $r_o$.

$$\boldsymbol{r_x = \sqrt{\frac{I_x}{A}}}, \quad \boldsymbol{r_y = \sqrt{\frac{I_y}{A}}}, \quad \boldsymbol{r_o = \sqrt{\frac{I_o}{A}}}$$

**Where:**

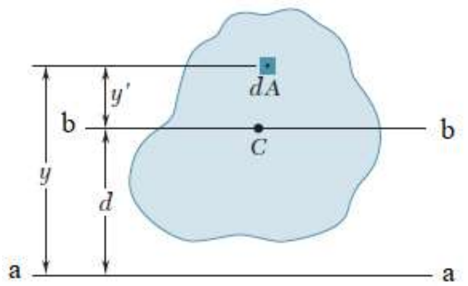
$r$: Radius of gyration, $I$: Moment of inertia, $A$: Cross – sectional area

## 7.3 The Parallel – Axis Theorem for Area

The moment of inertia of an area $A$ with respect to an axis $a - a$ can be determined from its moment of inertia with respect to the centroid axis $b - b$ by a calculation involving the distance $d$ between the axes.

$$dI_a = y^2 dA$$

$$dI_a = (\acute{y} + d_y)^2 dA$$




$$I_a = \int (\acute{y}^2 + 2\acute{y}\, d + d^2)\, dA$$

$$I_a = \int \acute{y}^2 dA + 2d \int \acute{y}\, dA + d^2 \int\ dA$$

$$I_a = I_C + 2d \int \acute{y}\, dA + A\, d^2$$

where, $\int \acute{y}\, dA$ is the first moment of are with respect to the $b - b$ axis. if the $b - b$ axis pass through the centroid of area, the expression $\int \acute{y}\, dA$ is zero, therefore:

$$I_a = I_C + A\, d^2$$

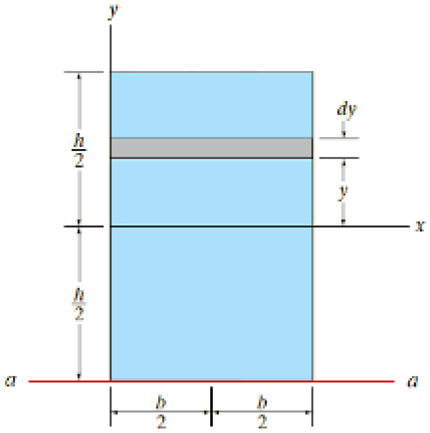
where:

$I_C$: is the second moment of the area with respect to an axis pass through the centroid of area and parallel to the $a - a$ – axis.

$d$: is the distance between an axis pass through the centroid of area to parallel $a - a$ – axis.

**Example No. 1:** Determine the moments of inertia of the rectangular area with respect to $a - a$ axis passing through the base of the rectangle.

**Solution:**




$$I_a = I_C + A\, d^2$$

$$I_C = I_x, \quad A = bh, \quad d = \frac{h}{2}$$

$$I_C = \int y^2 \; dA$$

$$dA = b\, dy$$

$$I_C = \int_{-h/2}^{h/2} y^2\, b\, dy = b\left[\frac{y^3}{3}\right]_{-h/2}^{h/2}$$

$$I_C = b\left[\frac{h^3}{24} - \frac{-h^3}{24}\right] = \frac{bh^3}{12}$$

$$\therefore I_a = \frac{bh^3}{12} + bh\left(\frac{h}{2}\right)^2$$

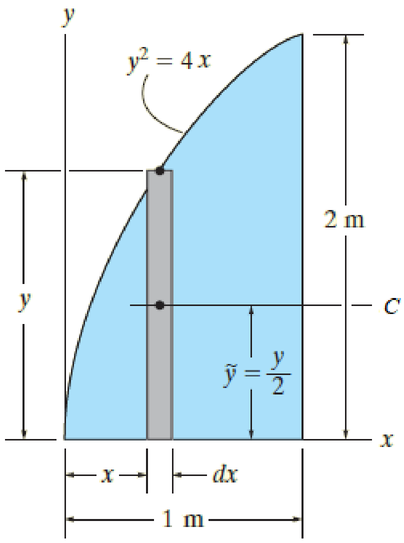
$$I_a = \frac{bh^3}{12} + \frac{bh^3}{4}$$

$$I_a = \frac{bh^3}{3}$$

**ملاحظة:** b هو البعد الموازي للمحور، h هو البعد العمودي على المحور.

**Example No. 2:** Determine the polar moments of inertia of the shaded area with respect to the axis through the origin, the equation of the curve is $y^2 = 4\,x$.

**Solution:**




$$I_o = I_y + I_x$$

$$I_y = \int dI_y = \int I_C + A\,d^2$$

$$I_y = \int_0^1 \left(\frac{bh^3}{12} + A\,d^2\right)$$

$$I_y = \int_0^1 \left(\frac{ydx^3}{12} + y\,dx \cdot x^2\right)$$

$$\frac{ydx^3}{12} \approx 0, \quad y = 2\,x^{1/2}$$

**ملاحظة:** $dx$ هو بعد جدا صغير لذلك فان تربيعه او تكعيبه يساوي صفر.

$$I_y = \int_0^1 2\,x^{1/2} \cdot x^2\;dx = 2\int_0^1 x^{5/2}\;dx = 2\left[\frac{x^{7/2}}{7/2}\right]_0^1 = 0.571\,m^4$$

$$I_x = \int dI_x = \int \frac{bh^3}{3}$$

**ملاحظة:** يطبق قانون $\frac{bh^3}{3}$ فقط اذا كان الشريحة منطبقة على المحور.

$$I_x = \int_0^1 \frac{y^3 dx}{3} = \int_0^1 \frac{8\,x^{3/2}}{3}dx = \frac{8}{3}\left[\frac{x^{5/2}}{5/2}\right]_0^1 = 1.067\,m^4$$

$$I_o = I_y + I_x = 0.571 + 1.067 = 1.638\,m^4$$

**H.W:** Solve by using a horizontal strip of area.