2.10. PROBLEMS INVOLVING TEMPERATURE CHANGES

All of the members and structures that we have considered so far were assumed to remain at the same temperature while they were being loaded. We are now going to consider various situations involving changes in temperature.

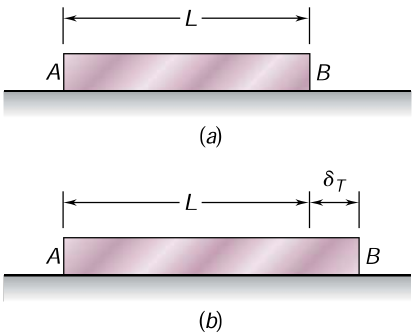


Fig. 2.34

Let us first consider a homogeneous rod *AB* of uniform cross section, which rests freely on a smooth horizontal surface (Fig. 2.34*a*). If the temperature of the rod is raised by ΔT, we observe that the rod elongates by an amount $\mathbf{d}_T$ which is proportional to both the temperature change ΔT and the length L of the rod (Fig. 2.34*b*). We have

$$\mathbf{d}_T = \mathbf{a}(\Delta T)L \qquad (2.21)$$

where $\mathbf{a}$ is a constant characteristic of the material, called the *coefficient of thermal expansion*. Since $\mathbf{d}_T$ and L are both expressed in units of length, $\mathbf{a}$ represents a quantity *per degree C,* or *per degree F,* depending whether the temperature change is expressed in degrees Celsius or in degrees Fahrenheit.

With the deformation $\mathbf{d}_T$ must be associated a strain $\epsilon_T = \mathbf{d}_T/L$. Recalling Eq. (2.21), we conclude that

$$\epsilon_T = \mathbf{a}\,\Delta T \qquad (2.22)$$

The strain ϵ_T is referred to as a *thermal strain,* since it is caused by the change in temperature of the rod. In the case we are considering here, there is *no stress associated with the strain* ϵ_T.

Let us now assume that the same rod *AB* of length L is placed between two fixed supports at a distance L from each other (Fig. 2.35*a*). Again, there is neither stress nor strain in this initial condition. If we raise the temperature by ΔT, the rod cannot elongate because of the restraints imposed on its ends; the elongation $\mathbf{d}_T$ of the rod is thus zero. Since the rod is homogeneous and of uniform cross section, the strain ϵ_T at any point is $\epsilon_T = \mathbf{d}_T/L$ and, thus, also zero. However, the supports will exert equal and opposite forces $\mathbf{P}$ and $\mathbf{P}'$ on the rod after the temperature has been raised, to keep it from elongating (Fig. 2.35*b*). It thus follows that a state of stress (with no corresponding strain) is created in the rod.

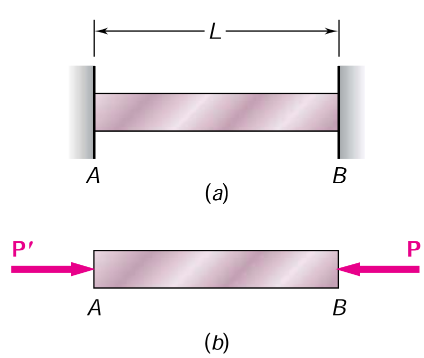


Fig. 2.35

As we prepare to determine the stress $\mathbf{s}$ created by the temperature change ΔT, we observe that the problem we have to solve is statically indeterminate. Therefore, we should first compute the magnitude P of the reactions at the supports from the condition that the elongation of the rod is zero. Using the superposition method described in Sec. 2.9, we detach the rod from its support B (Fig. 2.36*a*) and let it elongate freely as it undergoes the temperature change ΔT (Fig. 2.36*b*). According to formula (2.21), the corresponding elongation is

$$\mathbf{d}_T = \mathbf{a}(\Delta T)L$$

Applying now to end B the force $\mathbf{P}$ representing the redundant reaction, and recalling formula (2.7), we obtain a second deformation (Fig. 2.36*c*)

$$\mathbf{d}_P = \frac{PL}{AE}$$

Expressing that the total deformation $\mathbf{d}$ must be zero, we have

$$\mathbf{d} = \mathbf{d}_T + \mathbf{d}_P = \mathbf{a}(\Delta T)L + \frac{PL}{AE} = 0$$

from which we conclude that

$$P = -AE\mathbf{a}(\Delta T)$$

and that the stress in the rod due to the temperature change ΔT is

$$\mathbf{s} = \frac{P}{A} = -E\mathbf{a}(\Delta T) \qquad (2.23)$$

It should be kept in mind that the result we have obtained here and our earlier remark regarding the absence of any strain in the rod *apply only in the case of a homogeneous rod of uniform cross section*. Any other problem involving a restrained structure undergoing a change in temperature must be analyzed on its own merits. However, the same general approach can be used; i.e., we can consider separately the deformation due to the temperature change and the deformation due to the redundant reaction and superpose the solutions obtained.

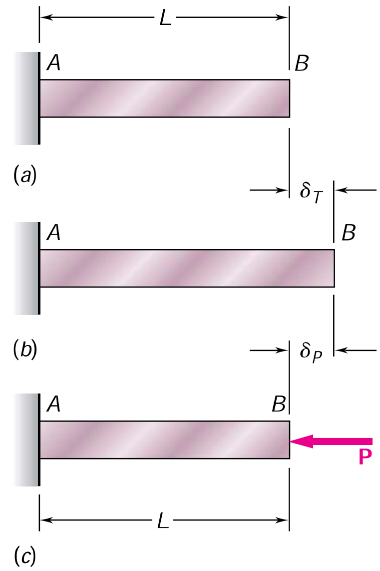


Fig. 2.36

EXAMPLE 2.06

Determine the values of the stress in portions AC and CB of the steel bar shown (Fig. 2.37) when the temperature of the bar is −50°F, knowing that a close fit exists at both of the rigid supports when the temperature is +75°F. Use the values $E = 29 \times 10^6$ psi and $\mathbf{a} = 6.5 \times 10^6/°F$ for steel.

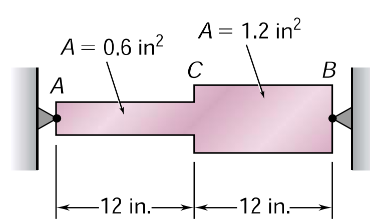


Fig. 2.37

We first determine the reactions at the supports. Since the problem is statically indeterminate, we detach the bar from its support at B and let it undergo the temperature change

$$\Delta T = (-50°F) - (75°F) = -125°F$$

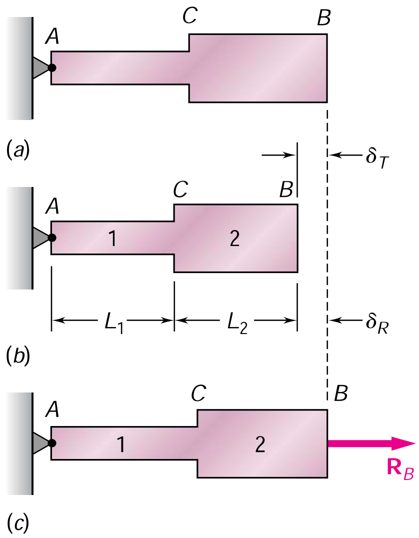


Fig. 2.38

The corresponding deformation (Fig. 2.38*b*) is

$$\begin{aligned}\delta_T &= \alpha(\Delta T)L = (6.5 \times 10^{-6}/°\text{F})(-125°\text{F})(24 \text{ in.})\\ &= -19.50 \times 10^{-3} \text{ in.}\end{aligned}$$

Applying now the unknown force $\mathbf{R}_B$ at end B (Fig. 2.38*c*), we use Eq. (2.8) to express the corresponding deformation δ_R. Substituting

$$L_1 = L_2 = 12 \text{ in.}$$
$$A_1 = 0.6 \text{ in}^2 \qquad A_2 = 1.2 \text{ in}^2$$
$$P_1 = P_2 = R_B \qquad E = 29 \times 10^6 \text{ psi}$$

into Eq. (2.8), we write

$$\begin{aligned}\delta_R &= \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E}\\ &= \frac{R_B}{29 \times 10^6 \text{ psi}}\left(\frac{12 \text{ in.}}{0.6 \text{ in}^2} + \frac{12 \text{ in.}}{1.2 \text{ in}^2}\right)\\ &= (1.0345 \times 10^{-6} \text{ in./lb})R_B\end{aligned}$$

Expressing that the total deformation of the bar must be zero as a result of the imposed constraints, we write

$$\begin{aligned}\delta &= \delta_T + \delta_R = 0\\ &= -19.50 \times 10^{-3} \text{ in.} + (1.0345 \times 10^{-6} \text{ in./lb})R_B = 0\end{aligned}$$

from which we obtain

$$R_B = 18.85 \times 10^3 \text{ lb} = 18.85 \text{ kips}$$

The reaction at A is equal and opposite.

Noting that the forces in the two portions of the bar are $P_1 = P_2 = 18.85$ kips, we obtain the following values of the stress in portions AC and CB of the bar:

$$\sigma_1 = \frac{P_1}{A_1} = \frac{18.85 \text{ kips}}{0.6 \text{ in}^2} = +31.42 \text{ ksi}$$
$$\sigma_2 = \frac{P_2}{A_2} = \frac{18.85 \text{ kips}}{1.2 \text{ in}^2} = +15.71 \text{ ksi}$$

We cannot emphasize too strongly the fact that, while the *total deformation* of the bar must be zero, the deformations of the portions AC and CB *are not zero*. A solution of the problem based on the assumption that these deformations are zero would therefore be wrong. Neither can the values of the strain in AC or CB be assumed equal to zero. To amplify this point, let us determine the strain ϵ_{AC} in portion AC of the bar. The strain ϵ_{AC} can be divided into two component parts; one is the thermal strain ϵ_T produced in the unrestrained bar by the temperature change ΔT (Fig. 2.38*b*). From Eq. (2.22) we write

$$\begin{aligned}\epsilon_T &= \alpha\, \Delta T = (6.5 \times 10^{-6}/°\text{F})(-125°\text{F})\\ &= -812.5 \times 10^{-6} \text{ in./in.}\end{aligned}$$

The other component of ϵ_{AC} is associated with the stress σ_1 due to the force $\mathbf{R}_B$ applied to the bar (Fig. 2.38*c*). From Hooke's law, we express this component of the strain as

$$\frac{\sigma_1}{E} = \frac{+31.42 \times 10^3 \text{ psi}}{29 \times 10^6 \text{ psi}} = +1083.4 \times 10^{-6} \text{ in./in.}$$

Adding the two components of the strain in AC, we obtain

$$\begin{aligned}\epsilon_{AC} &= \epsilon_T + \frac{\sigma_1}{E} = -812.5 \times 10^{-6} + 1083.4 \times 10^{-6}\\ &= +271 \times 10^{-6} \text{ in./in.}\end{aligned}$$

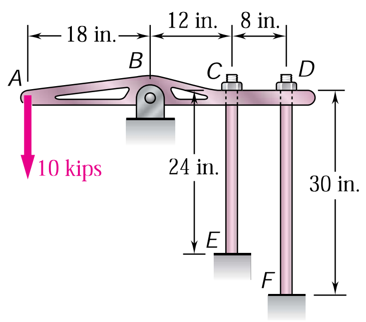
A similar computation yields the strain in portion CB of the bar:

$$\begin{aligned}\epsilon_{CB} &= \epsilon_T + \frac{\sigma_2}{E} = -812.5 \times 10^{-6} + 541.7 \times 10^{-6}\\ &= -271 \times 10^{-6} \text{ in./in.}\end{aligned}$$

The deformations δ_{AC} and δ_{CB} of the two portions of the bar are expressed respectively as

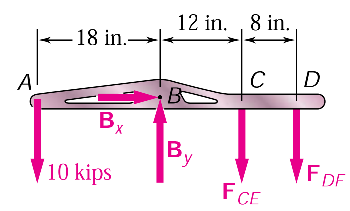
$$\begin{aligned}\delta_{AC} &= \epsilon_{AC}(AC) = (+271 \times 10^{-6})(12 \text{ in.})\\ &= +3.25 \times 10^{-3} \text{ in.}\\ \delta_{CB} &= \epsilon_{CB}(CB) = (-271 \times 10^{-6})(12 \text{ in.})\\ &= -3.25 \times 10^{-3} \text{ in.}\end{aligned}$$

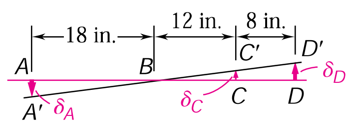
We thus check that, while the sum $\delta = \delta_{AC} + \delta_{CB}$ of the two deformations is zero, neither of the deformations is zero.

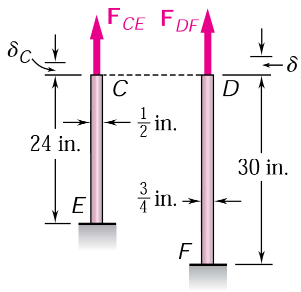


SAMPLE PROBLEM 2.3

The $\frac{1}{2}$-in.-diameter rod *CE* and the $\frac{3}{4}$-in.-diameter rod *DF* are attached to the rigid bar *ABCD* as shown. Knowing that the rods are made of aluminum and using $E = 10.6 \times 10^6$ psi, determine (*a*) the force in each rod caused by the loading shown, (*b*) the corresponding deflection of point *A*.







SOLUTION

Statics. Considering the free body of bar *ABCD*, we note that the reaction at *B* and the forces exerted by the rods are indeterminate. However, using statics, we may write

$$+\circlearrowleft \Sigma M_B = 0: \quad (10 \text{ kips})(18 \text{ in.}) - F_{CE}(12 \text{ in.}) - F_{DF}(20 \text{ in.}) = 0$$

$$12F_{CE} + 20F_{DF} = 180 \qquad (1)$$

Geometry. After application of the 10-kip load, the position of the bar is *A′BC′D′*. From the similar triangles *BAA′*, *BCC′*, and *BDD′* we have

$$\frac{\delta_C}{12 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_C = 0.6\delta_D \qquad (2)$$

$$\frac{\delta_A}{18 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_A = 0.9\delta_D \qquad (3)$$

Deformations. Using Eq. (2.7), we have

$$\delta_C = \frac{F_{CE}L_{CE}}{A_{CE}E} \qquad \delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E}$$

Substituting for δ_C and δ_D into (2), we write

$$\delta_C = 0.6\delta_D \qquad \frac{F_{CE}L_{CE}}{A_{CE}E} = 0.6\frac{F_{DF}L_{DF}}{A_{DF}E}$$

$$F_{CE} = 0.6\frac{L_{DF}}{L_{CE}}\frac{A_{CE}}{A_{DF}}F_{DF} = 0.6\left(\frac{30 \text{ in.}}{24 \text{ in.}}\right)\left[\frac{\frac{1}{4}\pi(\frac{1}{2} \text{ in.})^2}{\frac{1}{4}\pi(\frac{3}{4} \text{ in.})^2}\right]F_{DF} \qquad F_{CE} = 0.333F_{DF}$$

Force in Each Rod. Substituting for F_{CE} into (1) and recalling that all forces have been expressed in kips, we have

$$12(0.333F_{DF}) + 20F_{DF} = 180 \qquad F_{DF} = 7.50 \text{ kips} \blacktriangleleft$$

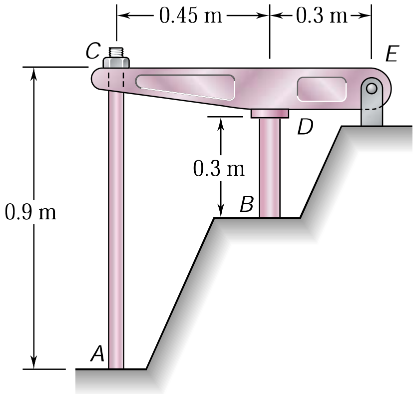
$$F_{CE} = 0.333F_{DF} = 0.333(7.50 \text{ kips}) \qquad F_{CE} = 2.50 \text{ kips} \blacktriangleleft$$

Deflections. The deflection of point *D* is

$$\delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E} = \frac{(7.50 \times 10^3 \text{ lb})(30 \text{ in.})}{\frac{1}{4}\pi(\frac{3}{4} \text{ in.})^2(10.6 \times 10^6 \text{ psi})} \qquad \delta_D = 48.0 \times 10^{-3} \text{ in.}$$

Using (3), we write

$$\delta_A = 0.9\delta_D = 0.9(48.0 \times 10^{-3} \text{ in.}) \qquad \delta_A = 43.2 \times 10^{-3} \text{ in.} \blacktriangleleft$$

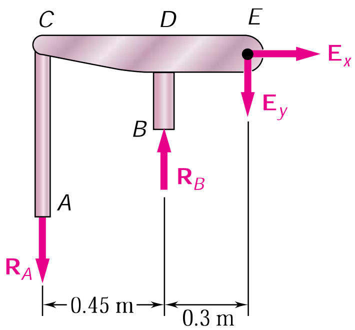


SAMPLE PROBLEM 2.4

The rigid bar *CDE* is attached to a pin support at *E* and rests on the 30-mm-diameter brass cylinder *BD*. A 22-mm-diameter steel rod *AC* passes through a hole in the bar and is secured by a nut which is snugly fitted when the temperature of the entire assembly is 20°C. The temperature of the brass cylinder is then raised to 50°C while the steel rod remains at 20°C. Assuming that no stresses were present before the temperature change, determine the stress in the cylinder.

Rod *AC*: Steel	Cylinder *BD*: Brass
$E = 200$ GPa	$E = 105$ GPa
$\alpha = 11.7 \times 10^{-6}/°C$	$\alpha = 20.9 \times 10^{-6}/°C$

SOLUTION



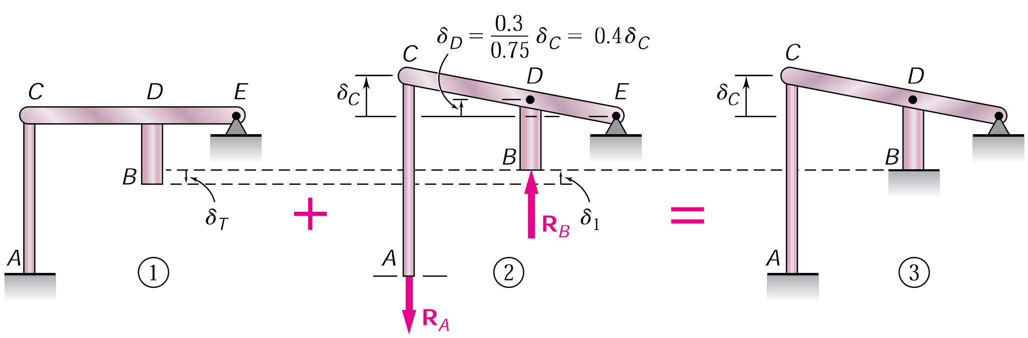
Statics. Considering the free body of the entire assembly, we write

$$+\circlearrowleft \Sigma M_E = 0: \quad R_A(0.75\text{ m}) - R_B(0.3\text{ m}) = 0 \qquad R_A = 0.4R_B \qquad (1)$$

Deformations. We use the method of superposition, considering $\mathbf{R}_B$ as redundant. With the support at *B* removed, the temperature rise of the cylinder causes point *B* to move down through δ_T. The reaction $\mathbf{R}_B$ must cause a deflection δ_1 equal to δ_T so that the final deflection of *B* will be zero (Fig. 3).

Deflection δ_T. Because of a temperature rise of 50° − 20° = 30°C, the length of the brass cylinder increases by δ_T.

$$\delta_T = L(\Delta T)\alpha = (0.3\text{ m})(30°C)(20.9 \times 10^{-6}/°C) = 188.1 \times 10^{-6}\text{ m}\downarrow$$



Deflection δ_1. We note that $\delta_D = 0.4\delta_C$ and $\delta_1 = \delta_D + \delta_{B/D}$.

$$\delta_C = \frac{R_A L}{AE} = \frac{R_A(0.9\text{ m})}{\frac{1}{4}\pi(0.022\text{ m})^2(200\text{ GPa})} = 11.84 \times 10^{-9}R_A\uparrow$$

$$\delta_D = 0.40\delta_C = 0.4(11.84 \times 10^{-9}R_A) = 4.74 \times 10^{-9}R_A\uparrow$$

$$\delta_{B/D} = \frac{R_B L}{AE} = \frac{R_B(0.3\text{ m})}{\frac{1}{4}\pi(0.03\text{ m})^2(105\text{ GPa})} = 4.04 \times 10^{-9}R_B\uparrow$$

We recall from (1) that $R_A = 0.4R_B$ and write

$$\delta_1 = \delta_D + \delta_{B/D} = [4.74(0.4R_B) + 4.04R_B]10^{-9} = 5.94 \times 10^{-9}R_B\uparrow$$

But $\delta_T = \delta_1$: $\quad 188.1 \times 10^{-6}\text{ m} = 5.94 \times 10^{-9}R_B \qquad R_B = 31.7$ kN

Stress in Cylinder: $\quad \sigma_B = \frac{R_B}{A} = \frac{31.7\text{ kN}}{\frac{1}{4}\pi(0.03)^2} \qquad \sigma_B = 44.8$ MPa ◀