2.9. STATICALLY INDETERMINATE PROBLEMS

In the problems considered in the preceding section, we could always use free-body diagrams and equilibrium equations to determine the internal forces produced in the various portions of a member under given loading conditions. The values obtained for the internal forces were then entered into Eq. (2.8) or (2.9) to obtain the deformation of the member.

There are many problems, however, in which the internal forces cannot be determined from statics alone. In fact, in most of these problems the reactions themselves—which are external forces—cannot be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relations involving deformations obtained by considering the geometry of the problem. Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are said to be *statically indeterminate*. The following examples will show how to handle this type of problem.

EXAMPLE 2.02

A rod of length L, cross-sectional area A_1, and modulus of elasticity E_1, has been placed inside a tube of the same length L, but of cross-sectional area A_2 and modulus of elasticity E_2 (Fig. 2.25*a*). What is the deformation of the rod and tube when a force **P** is exerted on a rigid end plate as shown?

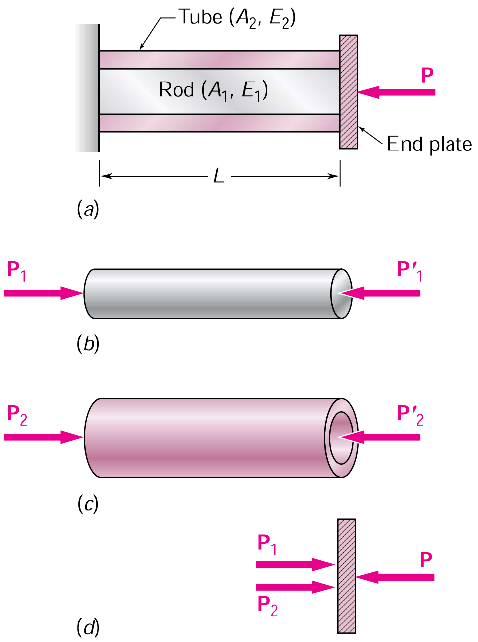

Fig. 2.25

Denoting by P_1 and P_2, respectively, the axial forces in the rod and in the tube, we draw free-body diagrams of all three elements (Fig. 2.25*b*, *c*, *d*). Only the last of the diagrams yields any significant information, namely:

$$P_1 + P_2 = P \tag{2.11}$$

Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2. The problem is statically indeterminate.

However, the geometry of the problem shows that the deformations δ_1 and δ_2 of the rod and tube must be equal. Recalling Eq. (2.7), we write

$$\delta_1 = \frac{P_1 L}{A_1 E_1} \qquad \delta_2 = \frac{P_2 L}{A_2 E_2} \tag{2.12}$$

Equating the deformations δ_1 and δ_2, we obtain:

$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \tag{2.13}$$

Equations (2.11) and (2.13) can be solved simultaneously for P_1 and P_2:

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2} \qquad P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$

Either of Eqs. (2.12) can then be used to determine the common deformation of the rod and tube.

EXAMPLE 2.03

A bar AB of length L and uniform cross section is attached to rigid supports at A and B before being loaded. What are the stresses in portions AC and BC due to the application of a load P at point C (Fig. 2.26*a*)?

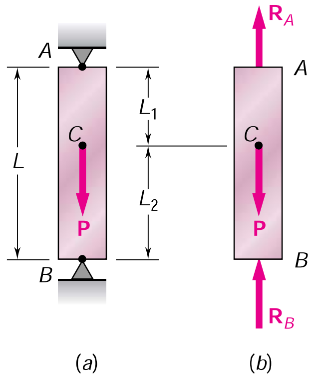

Fig. 2.26

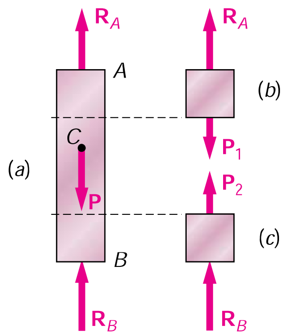

Fig. 2.27

Drawing the free-body diagram of the bar (Fig. 2.26*b*), we obtain the equilibrium equation

$$R_A + R_B = P \tag{2.14}$$

Since this equation is not sufficient to determine the two unknown reactions R_A and R_B, the problem is statically indeterminate.

However, the reactions may be determined if we observe from the geometry that the total elongation $\mathbf{d}$ of the bar must be zero. Denoting by $\mathbf{d}_1$ and $\mathbf{d}_2$, respectively, the elongations of the portions AC and BC, we write

$$\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2 = 0$$

or, expressing $\mathbf{d}_1$ and $\mathbf{d}_2$ in terms of the corresponding internal forces P_1 and P_2:

$$\mathbf{d} = \frac{P_1L_1}{AE} + \frac{P_2L_2}{AE} = 0 \tag{2.15}$$

But we note from the free-body diagrams shown respectively in parts b and c of Fig. 2.27 that $P_1 = R_A$ and $P_2 = -R_B$. Carrying these values into (2.15), we write

$$R_AL_1 - R_BL_2 = 0 \tag{2.16}$$

Equations (2.14) and (2.16) can be solved simultaneously for R_A and R_B; we obtain $R_A = PL_2/L$ and $R_B = PL_1/L$. The desired stresses $\mathbf{s}_1$ in AC and $\mathbf{s}_2$ in BC are obtained by dividing, respectively, $P_1 = R_A$ and $P_2 = -R_B$ by the cross-sectional area of the bar:

$$\mathbf{s}_1 = \frac{PL_2}{AL} \qquad \mathbf{s}_2 = -\frac{PL_1}{AL}$$

Superposition Method. We observe that a structure is statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium. This results in more unknown reactions than available equilibrium equations. It is often found convenient to designate one of the reactions as *redundant* and to eliminate the corresponding support. Since the stated conditions of the problem cannot be arbitrarily changed, the redundant reaction must be maintained in the solution. But it will be treated as an *unknown load* that, together with the other loads, must produce deformations that are compatible with the original constraints. The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction, and by adding—or *superposing*—the results obtained.†

†The general conditions under which the combined effect of several loads can be obtained in this way are discussed in Sec. 2.12.

EXAMPLE 2.04

Determine the reactions at A and B for the steel bar and loading shown in Fig. 2.28, assuming a close fit at both supports before the loads are applied.

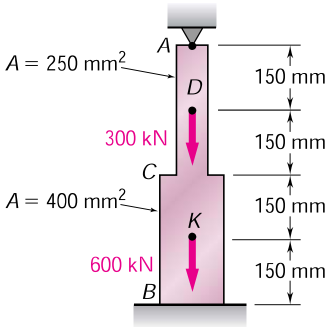

Fig. 2.28

We consider the reaction at B as redundant and release the bar from that support. The reaction $\mathbf{R}_B$ is now considered as an unknown load (Fig. 2.29a) and will be determined from the condition that the deformation $\boldsymbol{\delta}$ of the rod must be equal to zero. The solution is carried out by considering separately the deformation $\boldsymbol{\delta}_L$ caused by the given loads (Fig. 2.29b) and the deformation $\boldsymbol{\delta}_R$ due to the redundant reaction $\mathbf{R}_B$ (Fig. 2.29c).

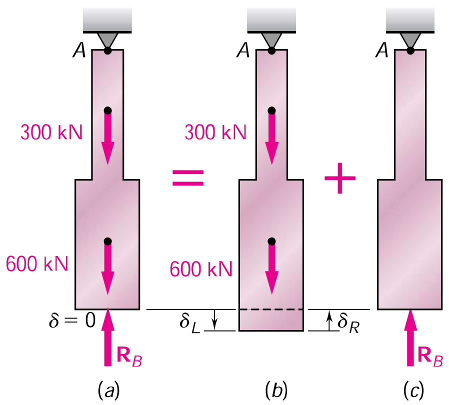

Fig. 2.29

The deformation $\boldsymbol{\delta}_L$ is obtained from Eq. (2.8) after the bar has been divided into four portions, as shown in Fig. 2.30.

Following the same procedure as in Example 2.01, we write

$$P_1 = 0 \quad P_2 = P_3 = 600 \times 10^3 \text{ N} \quad P_4 = 900 \times 10^3 \text{ N}$$
$$A_1 = A_2 = 400 \times 10^{-6} \text{ m}^2 \quad A_3 = A_4 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150 \text{ m}$$

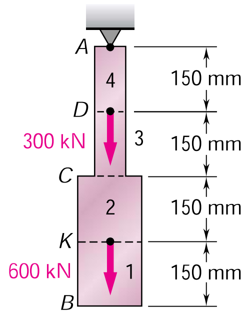

Fig. 2.30

Substituting these values into Eq. (2.8), we obtain

$$\boldsymbol{\delta}_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3 \text{ N}}{400 \times 10^{-6} \text{ m}^2} + \frac{600 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2} + \frac{900 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2}\right) \frac{0.150 \text{ m}}{E}$$

$$\boldsymbol{\delta}_L = \frac{1.125 \times 10^9}{E} \tag{2.17}$$

Considering now the deformation $\boldsymbol{\delta}_R$ due to the redundant reaction R_B, we divide the bar into two portions, as shown in Fig. 2.31, and write

$$P_1 = P_2 = -R_B$$
$$A_1 = 400 \times 10^{-6} \text{ m}^2 \quad A_2 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = 0.300 \text{ m}$$

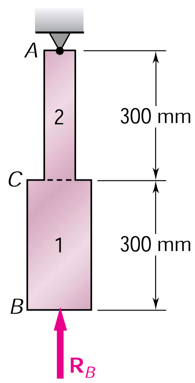

Fig. 2.31

Substituting these values into Eq. (2.8), we obtain

$$\mathbf{d}_R = \frac{P_1L_1}{A_1E} + \frac{P_2L_2}{A_2E} = -\frac{(1.95 \times 10^3)R_B}{E} \tag{2.18}$$

Expressing that the total deformation $\mathbf{d}$ of the bar must be zero, we write

$$\mathbf{d} = \mathbf{d}_L + \mathbf{d}_R = 0 \tag{2.19}$$

and, substituting for $\mathbf{d}_L$ and $\mathbf{d}_R$ from (2.17) and (2.18) into (2.19),

$$\mathbf{d} = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3)R_B}{E} = 0$$

Solving for R_B, we have

$$R_B = 577 \times 10^3\,\text{N} = 577\,\text{kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 2.32). We write

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300\,\text{kN} - 600\,\text{kN} + R_B = 0$$
$$R_A = 900\,\text{kN} - R_B = 900\,\text{kN} - 577\,\text{kN} = 323\,\text{kN}$$

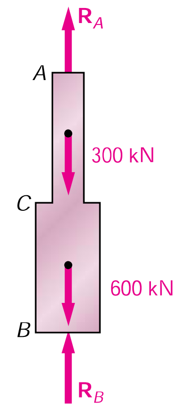

Fig. 2.32

Once the reactions have been determined, the stresses and strains in the bar can easily be obtained. It should be noted that, while the total deformation of the bar is zero, each of its component parts *does deform* under the given loading and restraining conditions.

EXAMPLE 2.05

Determine the reactions at A and B for the steel bar and loading of Example 2.04, assuming now that a 4.50-mm clearance exists between the bar and the ground before the loads are applied (Fig. 2.33). Assume $E = 200$ GPa.

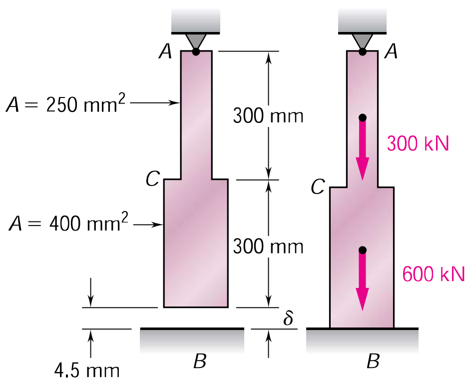

Fig. 2.33

We follow the same procedure as in Example 2.04. Considering the reaction at B as redundant, we compute the deformations $\mathbf{d}_L$ and $\mathbf{d}_R$ caused, respectively, by the given loads and by the redundant reaction $\mathbf{R}_B$. However, in this case the total deformation is not zero, but $\mathbf{d} = 4.5$ mm. We write therefore

$$\mathbf{d} = \mathbf{d}_L + \mathbf{d}_R = 4.5 \times 10^{-3}\,\text{m} \tag{2.20}$$

Substituting for $\mathbf{d}_L$ and $\mathbf{d}_R$ from (2.17) and (2.18) into (2.20), and recalling that $E = 200$ GPa $= 200 \times 10^9$ Pa, we have

$$\mathbf{d} = \frac{1.125 \times 10^9}{200 \times 10^9} - \frac{(1.95 \times 10^3)R_B}{200 \times 10^9} = 4.5 \times 10^{-3}\,\text{m}$$

Solving for R_B, we obtain

$$R_B = 115.4 \times 10^3\,\text{N} = 115.4\,\text{kN}$$

The reaction at A is obtained from the free-body diagram of the bar (Fig. 2.32):

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300\,\text{kN} - 600\,\text{kN} + R_B = 0$$
$$R_A = 900\,\text{kN} - R_B = 900\,\text{kN} - 115.4\,\text{kN} = 785\,\text{kN}$$