5.1. INTRODUCTION

This chapter and most of the next one will be devoted to the analysis and the design of *beams,* i.e., structural members supporting loads applied at various points along the member. Beams are usually long, straight prismatic members, as shown in the photo on the previous page. Steel and aluminum beams play an important part in both structural and mechanical engineering. Timber beams are widely used in home construction (Fig. 5.1). In most cases, the loads are perpendicular to the axis of the beam. Such a *transverse loading* causes only bending and shear in the beam. When the loads are not at a right angle to the beam, they also produce axial forces in the beam.

Fig. 5.1

The transverse loading of a beam may consist of *concentrated loads* $\mathbf{P}_1$, $\mathbf{P}_2$, . . . , expressed in newtons, pounds, or their multiples, kilonewtons and kips (Fig. 5.2*a*), of a *distributed load* w, expressed in N/m, kN/m, lb/ft, or kips/ft (Fig. 5.2*b*), or of a combination of both. When the load w per unit length has a constant value over part of the beam (as between A and B in Fig. 5.2*b*), the load is said to be *uniformly distributed* over that part of the beam.

Beams are classified according to the way in which they are supported. Several types of beams frequently used are shown in Fig. 5.3. The distance L shown in the various parts of the figure is called the *span*. Note that the reactions at the supports of the beams in parts *a, b,* and *c* of the figure involve a total of only three unknowns and, therefore, can be determined by

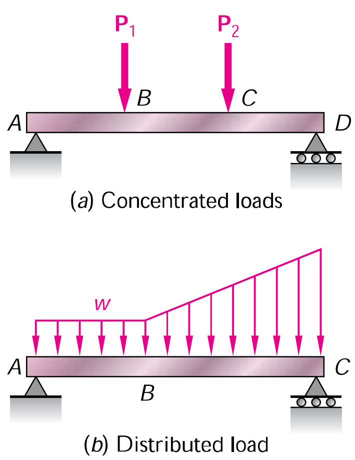


Fig. 5.2

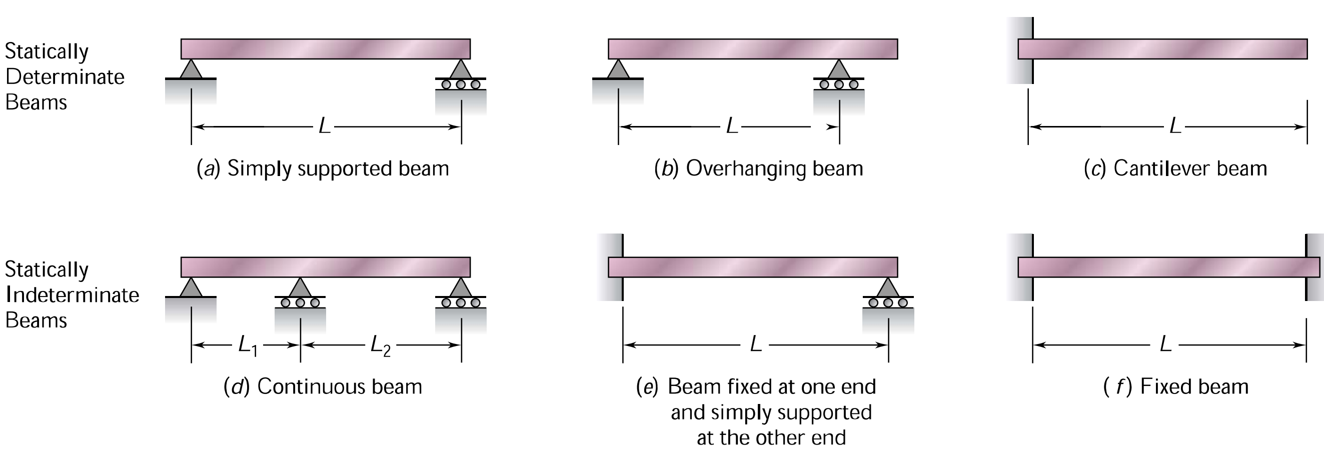


Fig. 5.3

the methods of statics. Such beams are said to be *statically determinate* and will be discussed in this chapter and the next. On the other hand, the reactions at the supports of the beams in parts *d, e,* and *f* of Fig. 5.3 involve more than three unknowns and cannot be determined by the methods of statics alone. The properties of the beams with regard to their resistance to deformations must be taken into consideration. Such beams are said to be *statically indeterminate* and their analysis will be postponed until Chap. 9, where deformations of beams will be discussed.

Sometimes two or more beams are connected by hinges to form a single continuous structure. Two examples of beams hinged at a point H are shown in Fig. 5.4. It will be noted that the reactions at the supports involve four unknowns and cannot be determined from the free-body diagram of the two-beam system. They can be determined, however, by considering the free-body diagram of each beam separately; six unknowns are involved (including two force components at the hinge), and six equations are available.

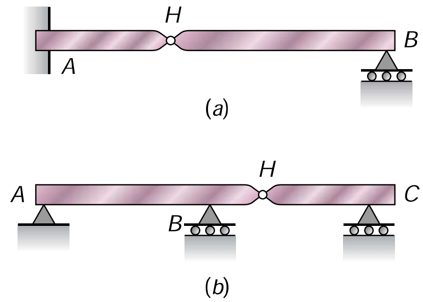


Fig. 5.4

It was shown in Sec. 4.1 that if we pass a section through a point C of a cantilever beam supporting a concentrated load **P** at its end (Fig. 4.6), the internal forces in the section are found to consist of a shear force **P**′ equal and opposite to the load **P** and a bending couple **M** of moment equal to the moment of **P** about C. A similar situation prevails for other types of supports and loadings. Consider, for example, a simply supported beam AB carrying two concentrated loads and a uniformly distributed load (Fig. 5.5*a*). To determine the internal forces in a section through point C we first draw the free-body diagram of the entire beam to obtain the reactions at the supports (Fig. 5.5*b*). Passing a section through C, we then draw the free-body diagram of AC (Fig. 5.5*c*), from which we determine the shear force **V** and the bending couple **M.**

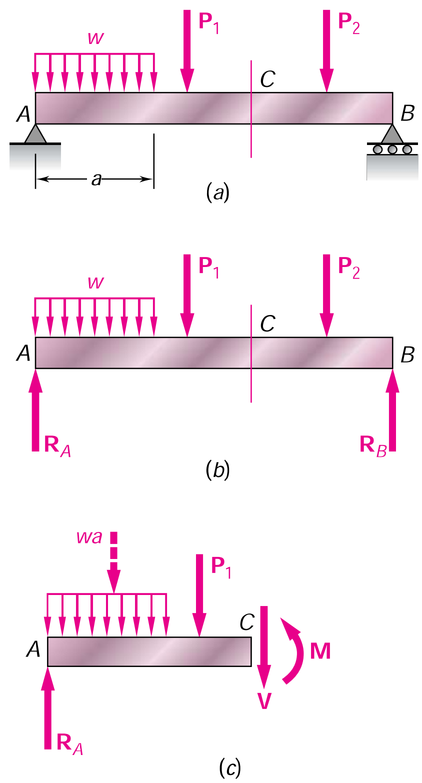


Fig. 5.5

The bending couple **M** creates *normal stresses* in the cross section, while the shear force **V** creates *shearing stresses* in that section. In most cases the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam. The determination of the normal stresses in a beam will be the subject of this chapter, while shearing stresses will be discussed in Chap. 6.

Since the distribution of the normal stresses in a given section depends only upon the value of the bending moment M in that section and the geometry of the section,† the elastic flexure formulas derived in Sec. 4.4 can be used to determine the maximum stress, as well as the stress at any given point, in the section. We write‡

$$\sigma_m = \frac{|M|c}{I} \qquad \sigma_x = -\frac{My}{I} \qquad (5.1, 5.2)$$

where I is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the couple, y is the distance from the neutral surface, and c is the maximum value of that distance (Fig. 4.13). We also recall from Sec. 4.4 that, introducing the

†It is assumed that the distribution of the normal stresses in a given cross section is not affected by the deformations caused by the shearing stresses. This assumption will be verified in Sec. 6.5.

‡We recall from Sec. 4.2 that M can be positive or negative, depending upon whether the concavity of the beam at the point considered faces upward or downward. Thus, in the case considered here of a transverse loading, the sign of M can vary along the beam. On the other hand, σ_m is a positive quantity, the absolute value of M is used in Eq. (5.1).

elastic section modulus $S = I/c$ of the beam, the maximum value σ_m of the normal stress in the section can be expressed as

$$\sigma_m = \frac{|M|}{S} \tag{5.3}$$

The fact that σ_m is inversely proportional to S underlines the importance of selecting beams with a large section modulus. Section moduli of various rolled-steel shapes are given in Appendix C, while the section modulus of a rectangular shape can be expressed, as shown in Sec. 4.4, as

$$S = \tfrac{1}{6}bh^2 \tag{5.4}$$

where b and h are, respectively, the width and the depth of the cross section.

Equation (5.3) also shows that, for a beam of uniform cross section, σ_m is proportional to $|M|$: Thus, the maximum value of the normal stress in the beam occurs in the section where $|M|$ is largest. It follows that one of the most important parts of the design of a beam for a given loading condition is the determination of the location and magnitude of the largest bending moment.

This task is made easier if a *bending-moment diagram* is drawn, i.e., if the value of the bending moment M is determined at various points of the beam and plotted against the distance x measured from one end of the beam. It is further facilitated if a *shear diagram* is drawn at the same time by plotting the shear V against x.

The sign convention to be used to record the values of the shear and bending moment will be discussed in Sec. 5.2. The values of V and M will then be obtained at various points of the beam by drawing free-body diagrams of successive portions of the beam. In Sec. 5.3 relations among load, shear, and bending moment will be derived and used to obtain the shear and bending-moment diagrams. This approach facilitates the determination of the largest absolute value of the bending moment and, thus, the determination of the maximum normal stress in the beam.

In Sec. 5.4 you will learn to design a beam for bending, i.e., so that the maximum normal stress in the beam will not exceed its allowable value. As indicated earlier, this is the dominant criterion in the design of a beam.

Another method for the determination of the maximum values of the shear and bending moment, based on expressing V and M in terms of *singularity functions,* will be discussed in Sec. 5.5. This approach lends itself well to the use of computers and will be expanded in Chap. 9 to facilitate the determination of the slope and deflection of beams.

Finally, the design of *nonprismatic beams,* i.e., beams with a variable cross section, will be discussed in Sec. 5.6. By selecting the shape and size of the variable cross section so that its elastic section modulus $S = I/c$ varies along the length of the beam in the same way as $|M|$, it is possible to design beams for which the maximum normal stress in each section is equal to the allowable stress of the material. Such beams are said to be of *constant strength*.

5.2. SHEAR AND BENDING-MOMENT DIAGRAMS

As indicated in Sec. 5.1, the determination of the maximum absolute values of the shear and of the bending moment in a beam are greatly facilitated if V and M are plotted against the distance x measured from one end of the beam. Besides, as you will see in Chap. 9, the knowledge of M as a function of x is essential to the determination of the deflection of a beam.

In the examples and sample problems of this section, the shear and bending-moment diagrams will be obtained by determining the values of V and M at selected points of the beam. These values will be found in the usual way, i.e., by passing a section through the point where they are to be determined (Fig. 5.6*a*) and considering the equilibrium of the portion of beam located on either side of the section (Fig. 5.6*b*). Since the shear forces $\mathbf{V}$ and $\mathbf{V}'$ have opposite senses, recording the shear at point C with an up or down arrow would be meaningless, unless we indicated at the same time which of the free bodies AC and CB we are considering. For this reason, the shear V will be recorded with a sign: a *plus sign* if the shearing forces are directed as shown in Fig. 5.6*b*, and a *minus sign* otherwise. A similar convention will apply for the bending moment M. It will be considered as positive if the bending couples are directed as shown in that figure, and negative otherwise.† Summarizing the sign conventions we have presented, we state:

The shear V and the bending moment M at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown in Fig. 5.7a.

These conventions can be more easily remembered if we note that

1. *The shear at any given point of a beam is positive when the* **external** *forces (loads and reactions) acting on the beam tend to shear off the beam at that point as indicated in Fig. 5.7b.*
2. *The bending moment at any given point of a beam is positive when the* **external** *forces acting on the beam tend to bend the beam at that point as indicated in Fig. 5.7c.*

It is also of help to note that the situation described in Fig. 5.7, in which the values of the shear and of the bending moment are positive, is precisely the situation that occurs in the left half of a simply supported beam carrying a single concentrated load at its midpoint. This particular case is fully discussed in the next example.

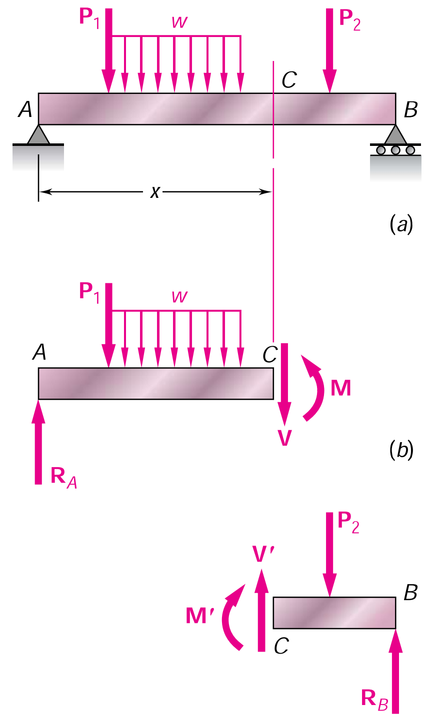
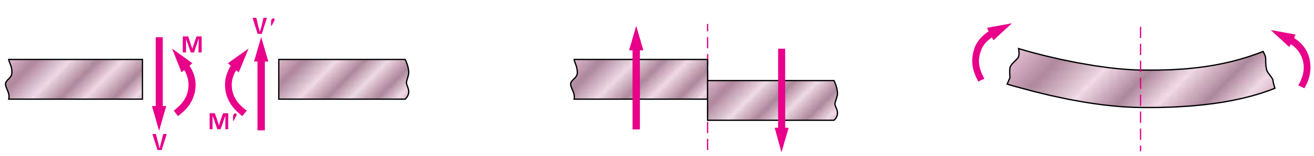


Fig. 5.6



(*a*) Internal forces (positive shear and positive bending moment)

(*b*) Effect of external forces (positive shear)

(*c*) Effect of external forces (positive bending moment)

Fig. 5.7

†Note that this convention is the same that we used earlier in Sec. 4.2

EXAMPLE 5.01

Draw the shear and bending-moment diagrams for a simply supported beam *AB* of span *L* subjected to a single concentrated load **P** at it midpoint *C* (Fig. 5.8).

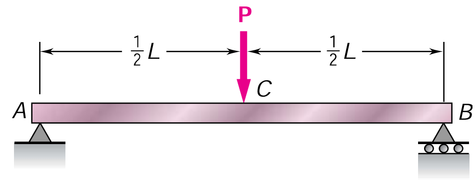


Fig. 5.8

We first determine the reactions at the supports from the free-body diagram of the entire beam (Fig. 5.9*a*); we find that the magnitude of each reaction is equal to $P/2$.

Next we cut the beam at a point *D* between *A* and *C* and draw the free-body diagrams of *AD* and *DB* (Fig. 5.9*b*). *Assuming that shear and bending moment are positive,* we direct the internal forces **V** and **V′** and the internal couples **M** and **M′** as indicated in Fig. 5.7*a*. Considering the free body *AD* and writing that the sum of the vertical components and the sum of the moments about *D* of the forces acting on the free body are zero, we find $V = +P/2$ and $M = +Px/2$. Both the shear and the bending moment are therefore positive; this may be checked by observing that the reaction at *A* tends to shear off and to bend the beam at *D* as indicated in Figs. 5.7*b* and *c*. We now plot *V* and *M* between *A* and *C* (Figs. 5.9*d* and *e*); the shear has a constant value $V = P/2$, while the bending moment increases linearly from $M = 0$ at $x = 0$ to $M = PL/4$ at $x = L/2$.

Cutting, now, the beam at a point *E* between *C* and *B* and considering the free body *EB* (Fig. 5.9*c*), we write that the sum of the vertical components and the sum of the moments about *E* of the forces acting on the free body are zero. We obtain $V = -P/2$ and $M = P(L - x)/2$. The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at *B* bends the beam at *E* as indicated in Fig. 5.7*c* but tends to shear it off in a manner opposite to that shown in Fig. 5.7*b*. We can complete, now, the shear and bending-moment diagrams of Figs. 5.9*d* and *e*; the shear has a constant value $V = -P/2$ between *C* and *B*, while the bending moment decreases linearly from $M = PL/4$ at $x = L/2$ to $M = 0$ at $x = L$.

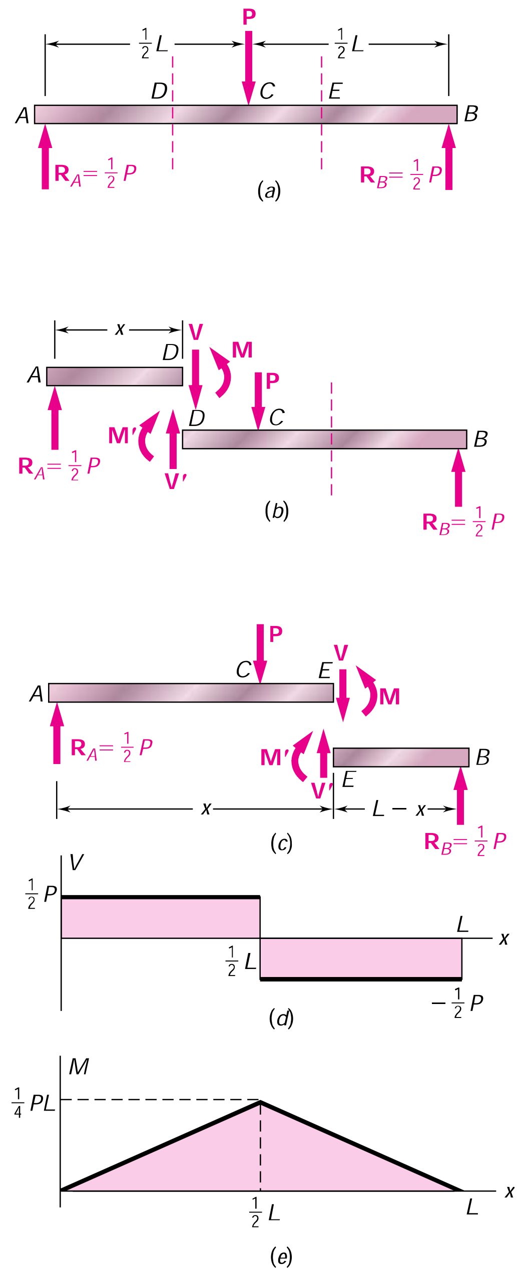


Fig. 5.9

We note from the foregoing example that, when a beam is subjected only to concentrated loads, the shear is constant between loads and the bending moment varies linearly between loads. In such situations, therefore, the shear and bending-moment diagrams can easily be drawn, once the values of V and M have been obtained at sections selected just to the left and just to the right of the points where the loads and reactions are applied (see Sample Prob. 5.1).

EXAMPLE 5.02

Draw the shear and bending-moment diagrams for a cantilever beam AB of span L supporting a uniformly distributed load w (Fig. 5.10).

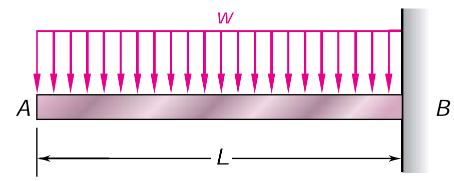


Fig. 5.10

We cut the beam at a point C between A and B and draw the free-body diagram of AC (Fig. 5.11a), directing **V** and **M** as indicated in Fig. 5.7a. Denoting by x the distance from A to C and replacing the distributed load over AC by its resultant wx applied at the midpoint of AC, we write

$$+\uparrow \Sigma F_y = 0: \qquad -wx - V = 0 \qquad V = -wx$$

$$+\circlearrowleft \Sigma M_C = 0: \qquad wx\left(\frac{x}{2}\right) + M = 0 \qquad M = -\frac{1}{2}wx^2$$

We note that the shear diagram is represented by an oblique straight line (Fig. 5.11b) and the bending-moment diagram by a parabola (Fig. 5.11c). The maximum values of V and M both occur at B, where we have

$$V_B = -wL \qquad M_B = -\tfrac{1}{2}wL^2$$

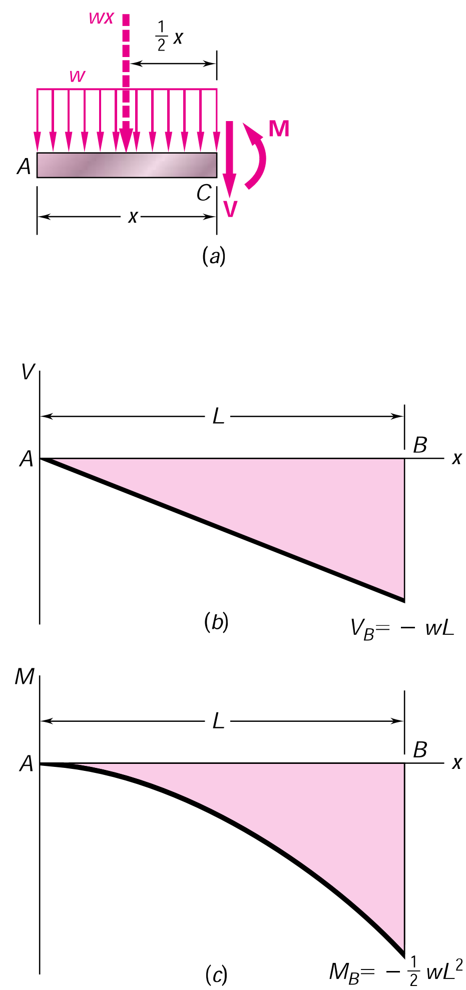
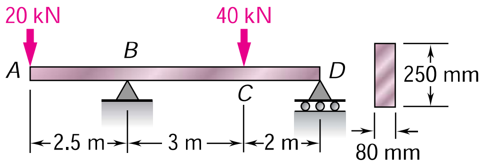
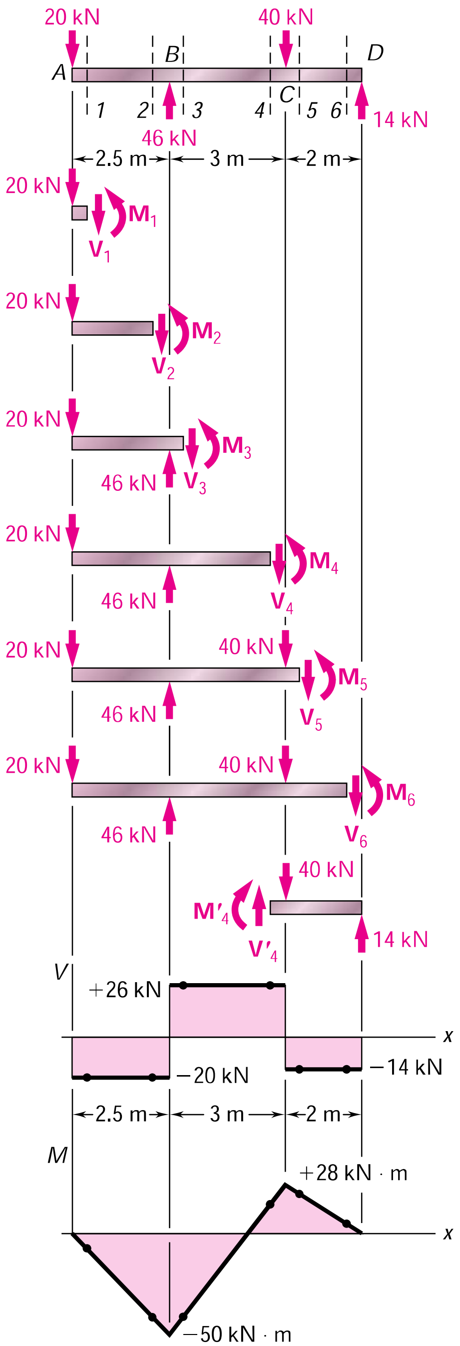


Fig. 5.11



SAMPLE PROBLEM 5.1

For the timber beam and loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.



SOLUTION

Reactions. Considering the entire beam as a free body, we find

$$\mathbf{R}_B = 40 \text{ kN}\uparrow \qquad \mathbf{R}_D = 14 \text{ kN}\uparrow$$

Shear and Bending-Moment Diagrams. We first determine the internal forces just to the right of the 20-kN load at A. Considering the stub of beam to the left of section *1* as a free body and assuming V and M to be positive (according to the standard convention), we write

$$+\uparrow\Sigma F_y = 0: \qquad -20 \text{ kN} - V_1 = 0 \qquad V_1 = -20 \text{ kN}$$
$$+\circlearrowleft\Sigma M_1 = 0: \qquad (20 \text{ kN})(0 \text{ m}) + M_1 = 0 \qquad M_1 = 0$$

We next consider as a free body the portion of beam to the left of section *2* and write

$$+\uparrow\Sigma F_y = 0: \qquad -20 \text{ kN} - V_2 = 0 \qquad V_2 = -20 \text{ kN}$$
$$+\circlearrowleft\Sigma M_2 = 0: \qquad (20 \text{ kN})(2.5 \text{ m}) + M_2 = 0 \qquad M_2 = -50 \text{ kN}\cdot\text{m}$$

The shear and bending moment at sections *3, 4, 5,* and *6* are determined in a similar way from the free-body diagrams shown. We obtain

$$V_3 = +26 \text{ kN} \qquad M_3 = -50 \text{ kN}\cdot\text{m}$$
$$V_4 = +26 \text{ kN} \qquad M_4 = +28 \text{ kN}\cdot\text{m}$$
$$V_5 = -14 \text{ kN} \qquad M_5 = +28 \text{ kN}\cdot\text{m}$$
$$V_6 = -14 \text{ kN} \qquad M_6 = 0$$

For several of the latter sections, the results may be more easily obtained by considering as a free body the portion of the beam to the right of the section. For example, for the portion of the beam to the right of section *4*, we have

$$+\uparrow\Sigma F_y = 0: \qquad V_4 - 40 \text{ kN} + 14 \text{ kN} = 0 \qquad V_4 = +26 \text{ kN}$$
$$+\circlearrowleft\Sigma M_4 = 0: \qquad -M_4 + (14 \text{ kN})(2 \text{ m}) = 0 \qquad M_4 = +28 \text{ kN}\cdot\text{m}$$

We can now plot the six points shown on the shear and bending-moment diagrams. As indicated earlier in this section, the shear is of constant value between concentrated loads, and the bending moment varies linearly; we obtain therefore the shear and bending-moment diagrams shown.

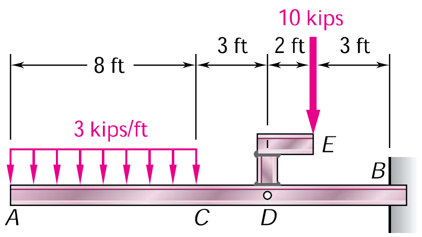
Maximum Normal Stress. It occurs at B, where $|M|$ is largest. We use Eq. (5.4) to determine the section modulus of the beam:

$$S = \tfrac{1}{6}bh^2 = \tfrac{1}{6}(0.080 \text{ m})(0.250 \text{ m})^2 = 833.33 \times 10^{-6} \text{ m}^3$$

Substituting this value and $|M| = |M_B| = 50 \times 10^3 \text{ N}\cdot\text{m}$ into Eq. (5.3):

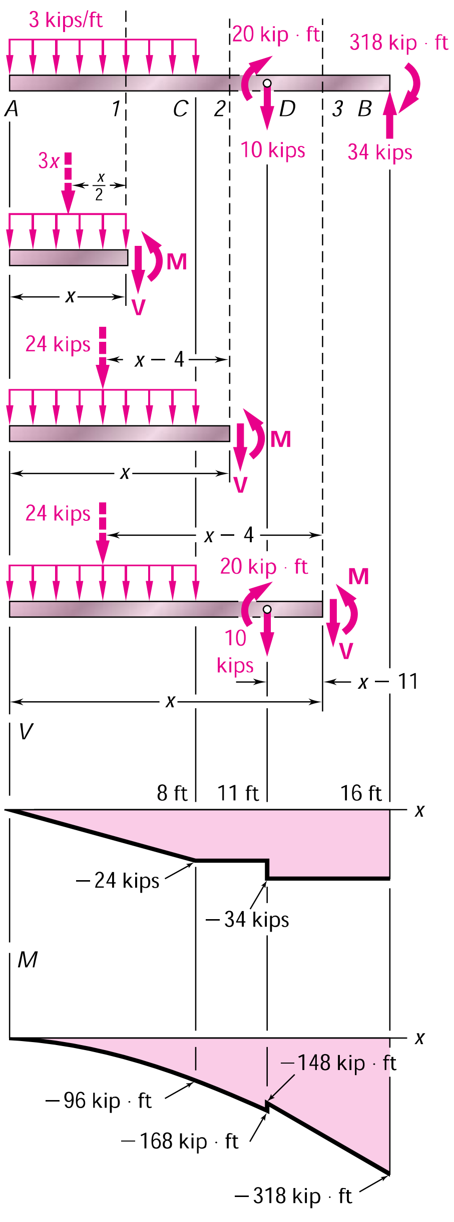
$$\sigma_m = \frac{|M_B|}{S} = \frac{(50 \times 10^3 \text{ N}\cdot\text{m})}{833.33 \times 10^{-6}} = 60.00 \times 10^6 \text{ Pa}$$

Maximum normal stress in the beam = 60.0 MPa ◄



SAMPLE PROBLEM 5.2

The structure shown consists of a W10 × 112 rolled-steel beam *AB* and of two short members welded together and to the beam. (*a*) Draw the shear and bending-moment diagrams for the beam and the given loading. (*b*) Determine the maximum normal stress in sections just to the left and just to the right of point *D*.



SOLUTION

Equivalent Loading of Beam. The 10-kip load is replaced by an equivalent force-couple system at *D*. The reaction at *B* is determined by considering the beam as a free body.

***a.* Shear and Bending-Moment Diagrams**

From A to C. We determine the internal forces at a distance x from point *A* by considering the portion of beam to the left of section *1*. That part of the distributed load acting on the free body is replaced by its resultant, and we write

$$+\uparrow \Sigma F_y = 0: \qquad -3x - V = 0 \qquad V = -3x \text{ kips}$$
$$+\circlearrowleft \Sigma M_1 = 0: \qquad 3x(\tfrac{1}{2}x) + M = 0 \qquad M = -1.5x^2 \text{ kip} \cdot \text{ft}$$

Since the free-body diagram shown can be used for all values of x smaller than 8 ft, the expressions obtained for V and M are valid in the region $0 < x < 8$ ft.

From C to D. Considering the portion of beam to the left of section *2* and again replacing the distributed load by its resultant, we obtain

$$+\uparrow \Sigma F_y = 0: \qquad -24 - V = 0 \qquad V = -24 \text{ kips}$$
$$+\circlearrowleft \Sigma M_2 = 0: \qquad 24(x-4) + M = 0 \qquad M = 96 - 24x \quad \text{kip} \cdot \text{ft}$$

These expressions are valid in the region 8 ft $< x <$ 11 ft.

From D to B. Using the position of beam to the left of section *3,* we obtain for the region 11 ft $< x <$ 16 ft

$$V = -34 \text{ kips} \qquad M = 226 - 34x \quad \text{kip} \cdot \text{ft}$$

The shear and bending-moment diagrams for the entire beam can now be plotted. We note that the couple of moment 20 kip · ft applied at point *D* introduces a discontinuity into the bending-moment diagram.

***b.* Maximum Normal Stress to the Left and Right of Point D.** From Appendix C we find that for the W10 × 112 rolled-steel shape, $S = 126 \text{ in}^3$ about the *X-X* axis.

To the left of D: We have $|M| = 168$ kip · ft = 2016 kip · in. Substituting for $|M|$ and S into Eq. (5.3), we write

$$\sigma_m = \frac{|M|}{S} = \frac{2016 \text{ kip} \cdot \text{in.}}{126 \text{ in}^3} = 16.00 \text{ ksi} \qquad \sigma_m = 16.00 \text{ ksi} \blacktriangleleft$$

To the right of D: We have $|M| = 148$ kip · ft = 1776 kip · in. Substituting for $|M|$ and S into Eq. (5.3), we write

$$\sigma_m = \frac{|M|}{S} = \frac{1776 \text{ kip} \cdot \text{in.}}{126 \text{ in}^3} = 14.10 \text{ ksi} \qquad \sigma_m = 14.10 \text{ ksi} \blacktriangleleft$$