7.9. STRESSES IN THIN-WALLED PRESSURE VESSELS

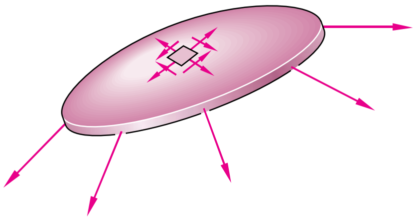


Fig. 7.48

Thin-walled pressure vessels provide an important application of the analysis of plane stress. Since their walls offer little resistance to bending, it can be assumed that the internal forces exerted on a given portion of wall are tangent to the surface of the vessel (Fig. 7.48). The resulting stresses on an element of wall will thus be contained in a plane tangent to the surface of the vessel.

Our analysis of stresses in thin-walled pressure vessels will be limited to the two types of vessels most frequently encountered: cylindrical pressure vessels and spherical pressure vessels (Figs. 7.49 and 7.50).

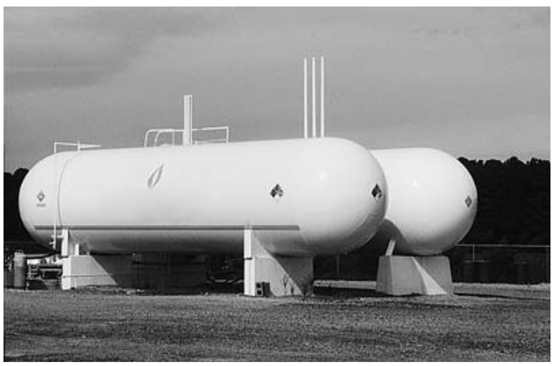


Fig. 7.49

Fig. 7.50

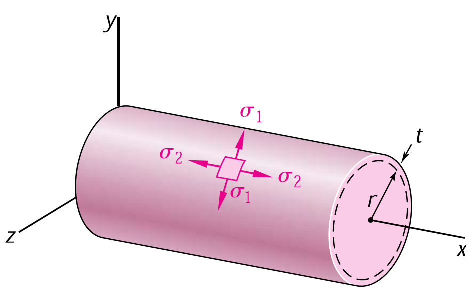


Fig. 7.51

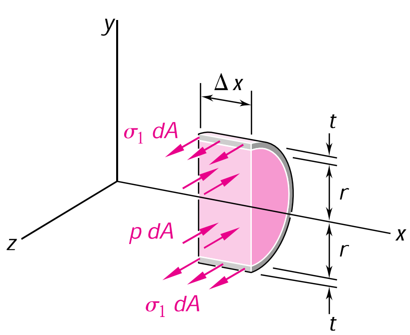


Fig. 7.52

Consider a cylindrical vessel of inner radius r and wall thickness t containing a fluid under pressure (Fig. 7.51). We propose to determine the stresses exerted on a small element of wall with sides respectively parallel and perpendicular to the axis of the cylinder. Because of the axisymmetry of the vessel and its contents, it is clear that no shearing stress is exerted on the element. The normal stresses σ_1 and σ_2 shown in Fig. 7.51 are therefore principal stresses. The stress σ_1 is known as the *hoop stress,* because it is the type of stress found in hoops used to hold together the various slats of a wooden barrel, and the stress σ_2 is called the *longitudinal stress*.

In order to determine the hoop stress σ_1, we detach a portion of the vessel and its contents bounded by the xy plane and by two planes parallel to the yz plane at a distance Δx from each other (Fig. 7.52). The forces parallel to the z axis acting on the free body defined in this fashion consist of the elementary internal forces $\sigma_1\, dA$ on the wall sections, and of the elementary pressure forces $p\, dA$ exerted on the portion of fluid included in the free body. Note that p denotes the *gage pressure* of the fluid, i.e., the excess of the inside pressure over the outside atmospheric pressure. The resultant of the internal forces $\sigma_1\, dA$ is equal to the product of σ_1 and of the cross-sectional area $2t\, \Delta x$ of the wall, while the resultant of the pressure forces $p\, dA$ is equal to the product of p and of the area $2r\, \Delta x$. Writing the equilibrium equation $\Sigma F_z = 0$, we have

$\Sigma F_z = 0$: $\qquad \sigma_1(2t\,\Delta x) - p(2r\,\Delta x) = 0$

and, solving for the hoop stress σ_1,

$$\sigma_1 = \frac{pr}{t} \tag{7.30}$$

To determine the longitudinal stress σ_2, we now pass a section perpendicular to the x axis and consider the free body consisting of the portion of the vessel and its contents located to the left of the section

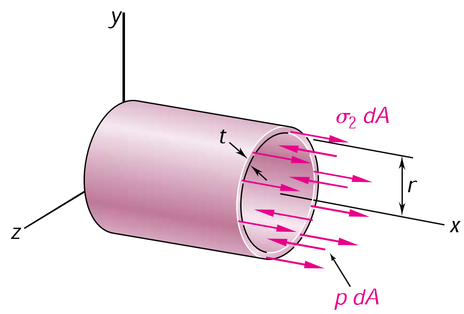


Fig. 7.53

(Fig. 7.53). The forces acting on this free body are the elementary internal forces $\sigma_2\, dA$ on the wall section and the elementary pressure forces $p\, dA$ exerted on the portion of fluid included in the free body. Noting that the area of the fluid section is πr^2 and that the area of the wall section can be obtained by multiplying the circumference $2\pi r$ of the cylinder by its wall thickness t, we write the equilibrium equation:†

$\Sigma F_x = 0$: $\qquad \sigma_2(2\pi r t) - p(\pi r^2) = 0$

and, solving for the longitudinal stress σ_2,

$$\sigma_2 = \frac{pr}{2t} \tag{7.31}$$

We note from Eqs. (7.30) and (7.31) that the hoop stress σ_1 is twice as large as the longitudinal stress σ_2:

$$\sigma_1 = 2\sigma_2 \tag{7.32}$$

†Using the mean radius of the wall section, $r_m = r + \frac{1}{2}t$, in computing the resultant of the forces on that section, we would obtain a more accurate value of the longitudinal stress, namely,

$$\sigma_2 = \frac{pr}{2t}\frac{1}{1 + \dfrac{t}{2r}} \tag{7.31$'$}$$

However, for a thin-walled pressure vessel, the term $t/2r$ is sufficiently small to allow the use of Eq. (7.31) for engineering design and analysis. If a pressure vessel is not thin-walled (i.e., if $t/2r$ is not small), the stresses σ_1 and σ_2 vary across the wall and must be determined by the methods of the theory of elasticity.

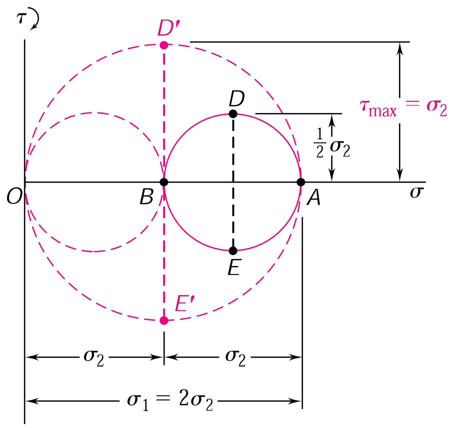


Fig. 7.54

Drawing Mohr's circle through the points A and B that correspond respectively to the principal stresses σ_1 and σ_2 (Fig. 7.54), and recalling that the maximum in-plane shearing stress is equal to the radius of this circle, we have

$$\tau_{max(in\ plane)} = \tfrac{1}{2}\sigma_2 = \frac{pr}{4t} \tag{7.33}$$

This stress corresponds to points D and E and is exerted on an element obtained by rotating the original element of Fig. 7.51 through 45° *within the plane* tangent to the surface of the vessel. The maximum shearing stress in the wall of the vessel, however, is larger. It is equal to the radius of the circle of diameter OA and corresponds to a rotation of 45° about a longitudinal axis and *out of the plane* of stress.† We have

$$\boxed{\tau_{max} = \sigma_2 = \frac{pr}{2t}} \tag{7.34}$$

We now consider a spherical vessel of inner radius r and wall thickness t, containing a fluid under a gage pressure p. For reasons of symmetry, the stresses exerted on the four faces of a small element of wall must be equal (Fig. 7.55). We have

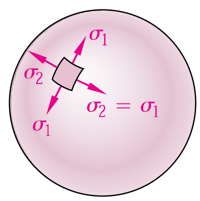


Fig. 7.55

$$\sigma_1 = \sigma_2 \tag{7.35}$$

To determine the value of the stress, we pass a section through the center C of the vessel and consider the free body consisting of the portion of the vessel and its contents located to the left of the section (Fig. 7.56). The equation of equilibrium for this free body is the same as for the free body of Fig. 7.53. We thus conclude that, for a spherical vessel,

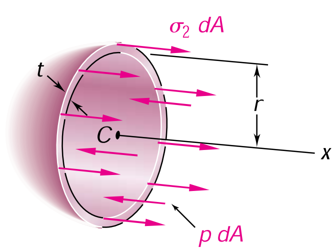


Fig. 7.56

$$\boxed{\sigma_1 = \sigma_2 = \frac{pr}{2t}} \tag{7.36}$$

Since the principal stresses σ_1 and σ_2 are equal, Mohr's circle for transformations of stress within the plane tangent to the surface of the vessel reduces to a point (Fig. 7.57); we conclude that the in-plane normal stress is constant and that the in-plane maximum shearing stress is zero. The maximum shearing stress in the wall of the vessel, however, is not zero; it is equal to the radius of the circle of diameter OA and corresponds to a rotation of 45° out of the plane of stress. We have

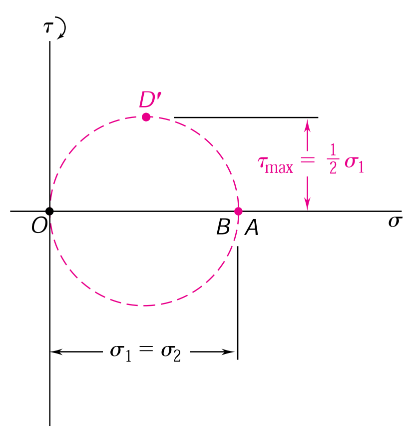


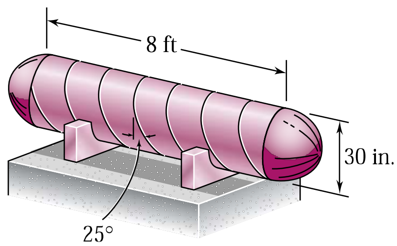
Fig. 7.57

$$\boxed{\tau_{max} = \tfrac{1}{2}\sigma_1 = \frac{pr}{4t}} \tag{7.37}$$

†It should be observed that, while the third principal stress is zero on the outer surface of the vessel, it is equal to $-p$ on the inner surface, and is represented by a point $C(-p, 0)$ on a Mohr-circle diagram. Thus, close to the inside surface of the vessel, the maximum shearing stress is equal to the radius of a circle of diameter CA, and we have

$$\tau_{max} = \frac{1}{2}(\sigma_1 + p) = \frac{pr}{2t}\left(1 + \frac{t}{r}\right)$$

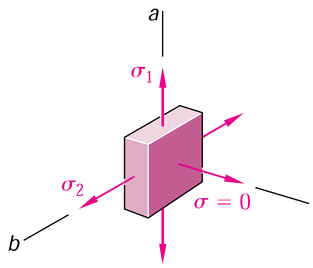
For a thin-walled vessel, however, the term t/r is small, and we can neglect the variation of τ_{max} across the wall section. This remark also applies to spherical pressure vessels.



SAMPLE PROBLEM 7.5

A compressed-air tank is supported by two cradles as shown; one of the cradles is designed so that it does not exert any longitudinal force on the tank. The cylindrical body of the tank has a 30-in. outer diameter and is fabricated from a $\frac{3}{8}$-in. steel plate by butt welding along a helix that forms an angle of 25° with a transverse plane. The end caps are spherical and have a uniform wall thickness of $\frac{5}{16}$ in. For an internal gage pressure of 180 psi, determine (*a*) the normal stress and the maximum shearing stress in the spherical caps, (*b*) the stresses in directions perpendicular and parallel to the helical weld.

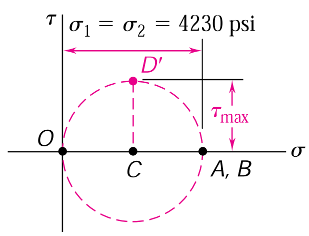
SOLUTION



***a*. Spherical Cap.** Using Eq. (7.36), we write

$$p = 180 \text{ psi}, t = \tfrac{5}{16} \text{ in.} = 0.3125 \text{ in.}, r = 15 - 0.3125 = 14.688 \text{ in.}$$

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{(180 \text{ psi})(14.688 \text{ in.})}{2(0.3125 \text{ in.})} \qquad \sigma = 4230 \text{ psi} \blacktriangleleft$$



We note that for stresses in a plane tangent to the cap, Mohr's circle reduces to a point (*A, B*) on the horizontal axis and that all in-plane shearing stresses are zero. On the surface of the cap the third principal stress is zero and corresponds to point *O*. On a Mohr's circle of diameter *AO,* point *D′* represents the maximum shearing stress; it occurs on planes at 45° to the plane tangent to the cap.

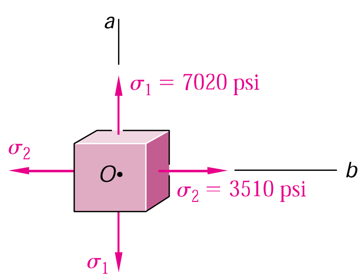
$$\tau_{max} = \tfrac{1}{2}(4230 \text{ psi}) \qquad \tau_{max} = 2115 \text{ psi} \blacktriangleleft$$

***b*. Cylindrical Body of the Tank.** We first determine the hoop stress σ_1 and the longitudinal stress σ_2. Using Eqs. (7.30) and (7.32), we write

$$p = 180 \text{ psi}, t = \tfrac{3}{8} \text{ in.} = 0.375 \text{ in.}, r = 15 - 0.375 = 14.625 \text{ in.}$$

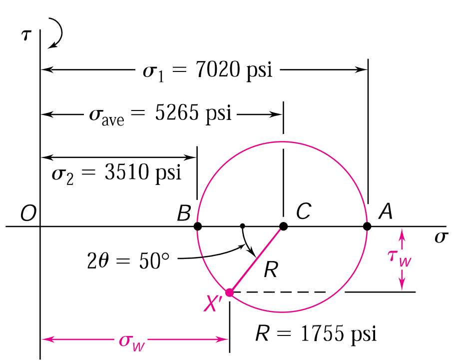
$$\sigma_1 = \frac{pr}{t} = \frac{(180 \text{ psi})(14.625 \text{ in.})}{0.375 \text{ in.}} = 7020 \text{ psi} \qquad \sigma_2 = \tfrac{1}{2}\sigma_1 = 3510 \text{ psi}$$

$$\sigma_{ave} = \tfrac{1}{2}(\sigma_1 + \sigma_2) = 5265 \text{ psi} \qquad R = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 1755 \text{ psi}$$



Stresses at the Weld. Noting that both the hoop stress and the longitudinal stress are principal stresses, we draw Mohr's circle as shown.

An element having a face parallel to the weld is obtained by rotating the face perpendicular to the axis *Ob* counterclockwise through 25°. Therefore, on Mohr's circle we locate the point *X′* corresponding to the stress components on the weld by rotating radius *CB* counterclockwise through $2\theta = 50°$.



$$\sigma_w = \sigma_{ave} - R\cos 50° = 5265 - 1755 \cos 50° \qquad \sigma_w = +4140 \text{ psi} \blacktriangleleft$$

$$\tau_w = R \sin 50° = 1755 \sin 50° \qquad \tau_w = 1344 \text{ psi} \blacktriangleleft$$

Since *X′* is below the horizontal axis, τ_w tends to rotate the element counterclockwise.