Sol:

$$a)\frac{dy}{dx} = -\frac{1}{1+\frac{4}{x^2}}\cdot\left(-\frac{2}{x^2}\right) + \frac{1}{1+\frac{x^2}{4}}\cdot\left(\frac{1}{2}\right) = \frac{4}{4+x^2}$$

$$b)\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2} = \frac{1}{(x+1)\sqrt{x}}$$

$$c)\frac{dy}{dx} = x\,\frac{-2}{\sqrt{1-4x^2}} + \cos^{-1}2x - \frac{1}{4}\cdot\frac{-8x}{\sqrt{1-4x^2}} = \cos^{-1}2x$$

$$d)\frac{dy}{dx} = \frac{5}{|5x|\sqrt{25x^2-1}} = \frac{1}{|x|\sqrt{25x^2-1}}$$

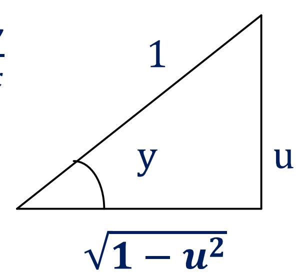
Ex 12: *Prove that*:

$$a)\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad b)\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

Proof: *a*)

$$Let\ y = \sin^{-1}u \Rightarrow u = \sin y \Rightarrow \frac{du}{dx} = \cos y\frac{dy}{dx} = \sqrt{1-u^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \Rightarrow \frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$



Hyperbolic functions: If u is any differentiable function of x, then:

21) $\frac{d}{dx}\sinh u = \cosh u\cdot\frac{du}{dx}$

22) $\frac{d}{dx}\cosh u = \sinh u\cdot\frac{du}{dx}$

23) $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u\cdot\frac{du}{dx}$

24) $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u\cdot\frac{du}{dx}$

25) $\frac{d}{dx}\operatorname{sech} u = -sech\,u\cdot\tanh u\cdot\frac{du}{dx}$

26) $\frac{d}{dx}\operatorname{csch} u = -csch\,u\cdot\coth u\cdot\frac{du}{dx}$

$Ex\ 13: Find\ \frac{dy}{dx}\ for\ the\ following\ functions:$

$$a)\ y = \coth(\tan x) \qquad b)\ y = \sin^{-1}(\tanh x) \qquad c)\ y = \ln\left|\tanh\frac{x}{2}\right|$$

$$d)\ y = x\cdot\sinh 2x - \frac{1}{2}\cdot\cosh 2x \qquad e)\ y = \operatorname{sech}^3 x \qquad f)\ y = \operatorname{csch}^2 x$$

$Sol:$

$$a)\ \frac{dy}{dx} = -\operatorname{csch}^2(\tan x)\cdot\sec^2 x$$

$$b)\ \frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \operatorname{sech} x$$

$$c)\ \frac{dy}{dx} = \frac{1}{\tanh\frac{x}{2}}\operatorname{sech}^2\frac{x}{2}\cdot\frac{1}{2} = \frac{\frac{1}{\cosh^2 x/2}}{2\cdot\frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}}$$

$$= \frac{1}{2\sinh\frac{x}{2}\cdot\cosh\frac{x}{2}} = \frac{1}{\sinh x} = \operatorname{csch} x$$

$Ex\ 14: Show\ that\ the\ functions:$

$$x = -\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} \quad and \quad y = \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \cosh\frac{t}{\sqrt{3}}$$

$Taken\ together, satisfy\ the\ differential\ equations:$

$$i)\ \frac{dx}{dt} + 2\frac{dy}{dt} + x = 0 \quad and \quad ii)\ \frac{dx}{dt} - \frac{dy}{dt} + y = 0$$

$Proof:$

$$x = -\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} \Rightarrow \frac{dx}{dt} = -\frac{2}{3}\cosh\frac{t}{\sqrt{3}}$$

$$y=\frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}}+\cosh\frac{t}{\sqrt{3}}\Rightarrow\frac{dy}{dt}=\frac{1}{3}\cosh\frac{t}{\sqrt{3}}+\frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}}$$

$$i)\frac{dx}{dt}+2\frac{dy}{dt}+x=-\frac{2}{3}\cosh\frac{t}{\sqrt{3}}+\frac{2}{3}\cosh\frac{t}{\sqrt{3}}+\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}}-\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}}=0$$

$$ii)\frac{dx}{dt}-\frac{dy}{dt}+y=\cdots\cdots$$

Ex 15: Prove that :

$$a)\frac{d}{dx}\tanh u=\operatorname{sech}^2 u\cdot\frac{du}{dx}\quad and\quad b)\frac{d}{dx}\operatorname{sech} u=-\operatorname{sech} u\cdot\tanh u\cdot\frac{du}{dx}$$

Proof:

$$a)\frac{d}{dx}\tanh u=\frac{d}{dx}\left(\frac{\sinh u}{\cosh u}\right)=\frac{\cosh u\cdot\cosh u\cdot\frac{du}{dx}-\sinh u\cdot\sinh u\cdot\frac{du}{dx}}{\cosh^2 u}$$

$$=\frac{(\cosh^2 u-\sinh^2 u)\frac{du}{dx}}{\cosh^2 u}=\frac{1}{\cosh^2 u}\cdot\frac{du}{dx}=\operatorname{sech}^2 u\cdot\frac{du}{dx}$$

$$b)\frac{d}{dx}\frac{1}{\cosh u}=-\frac{1}{\cosh^2 u}\cdot\sinh u\cdot\frac{du}{dx}=-\operatorname{sech} u\cdot\tanh u\cdot\frac{du}{dx}$$

Inverse hyperbolic functions: If u is any differentiable function of x, then:

$$27)\frac{d}{dx}\sinh^{-1}u=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}\qquad 28)\frac{d}{dx}\cosh^{-1}u=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

$$29)\frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}\frac{du}{dx}\quad |u|<1\qquad 30)\frac{d}{dx}\coth^{-1}u=\frac{1}{1-u^2}\frac{du}{dx}\quad |u|>1$$

$$31)\frac{d}{dx}\operatorname{sech}^{-1}u=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}\qquad 32)\frac{d}{dx}\operatorname{csch}^{-1}u=-\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$$

$Ex\ 16; Find \frac{dy}{dx}\ for\ the\ following\ functions:$

$$a)\ y = \cosh^{-1}(\sec x) \qquad b)\ y = \tanh^{-1}(\cos x)$$

$$c)\ y = \coth^{-1}(\sec x) \qquad d)\ y = \operatorname{sech}^{-1}(\sin 2x)$$

$Sol:$

$$a)\frac{dy}{dx} = \frac{sec\ x \cdot \tan x}{\sqrt{\sec^2 x - 1}} = \frac{sec\ x \cdot \tan x}{\sqrt{\tan^2 x}} = \sec x \qquad where \tan x > 0$$

$$b)\frac{dy}{dx} = -\frac{sinx}{1 - \cos^2 x} = \frac{-\sin x}{\sin^2 x} = -\csc x$$

$$c)\frac{dy}{dx} = \frac{\sec x \cdot \tan x}{1 - \sec^2 x} = \frac{\sec x \tan x}{-\tan^2 x} = -\csc x$$

$$d)\frac{dy}{dx} = -\frac{2\cos 2x}{\sin 2x \cdot \sqrt{1 - \sin^2 2x}} = -2\csc 2x \quad where \cos 2x > 0$$

$Ex\ 17: Verify\ the\ following\ formulas:$

$$a)\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx} \qquad b)\frac{d}{dx}\tanh^{-1} u = \frac{1}{1 - u^2}\frac{du}{dx} \quad |u| < 1$$

$Proof:$

$$a)\ Let\ y = \cosh^{-1} u \ \Rightarrow u = \cosh y \ \Rightarrow \frac{du}{dx}\sinh y \frac{dy}{dx} \ \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}\frac{du}{dx}$$

$$\cosh^2 y - \sinh^2 y = 1 \ \Rightarrow u^2 - \sinh^2 y = 1 \Rightarrow \sinh y = \sqrt{u^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx} \ \Rightarrow \frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}$$

The derivatives of functions like u^v:

$Where\ u\ and\ v\ are\ differentiable\ functions\ of\ x, are\ found\ by$

$logarithmic\ differentiation:$