$Ex7: Find\ \frac{dy}{dx}\ for\ the\ following\ functions:$

$a)\ y = 2^{3x}$ $\quad b)\ y = 2^x \cdot 3^x$ $\quad c)\ y = (2^x)^2$

$d)\ y = x \cdot 2^{x^2}$ $\quad e)\ y = e^{(x+e^{5x})}$ $\quad f)\ y = e^{\sqrt{1+5x^2}}$

$Sol:$

$a)\ y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} \cdot (\ln 2) \cdot 3$

$b)\ y = 2^x \cdot 3^x = 6^x \Rightarrow \frac{dy}{dx} = 6^x \ln 6$

$c)\ y = (2^x)^2 = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x}(\ln 2) \cdot 2 = 2^{2x+1} \ln 2$

$d)\ y = x \cdot 2^{x^2} \Rightarrow \frac{dy}{dx} = x \cdot 2^{x^2} \ln 2 \cdot 2x + 2^{x^2} = 2^{x^2}(2x^2 \ln 2 + 1)$

$e)\ y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})}(1 + 5e^{5x})$

$$f)\ y = e^{\sqrt{1+5x^2}} = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}} \frac{1}{2}(1 + 5x^2)^{-\frac{1}{2}} \cdot 10x$$

$= e^{\sqrt{1+5x^2}}\ \frac{5x}{\sqrt{1+5x^2}}$

<u>**Logarithm functions**</u>: $If\ u\ is\ any\ differentiable\ function\ of\ x, then$

$8)\ \frac{d}{dx} \log_a u = \frac{1}{u \cdot \ln a} \cdot \frac{du}{dx}$ $\quad and \quad$ $\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$

$Ex\ 8: Find\ \frac{dy}{dx}\ for\ the\ following\ functions:$

$a)\ y = \log_{10} e^x$ $\quad b)\ y = \log_5 (x + 1)^2$ $\quad c)\ y = \log_2 (3x^2 + 1)^3$

$d)\ y = [\ln(x^2 + 2)^2]^3$ $\quad e)\ y + \ln(xy) = 1$ $\quad f)\ y = \frac{(2x^3 - 4)^{\frac{2}{3}} \cdot (2x^2 + 3)^{\frac{5}{2}}}{(7x^3 + 4x - 3)^2}$

$Sol:$

$$a)\ y=\log_{10} e^{x} \Rightarrow \frac{1}{e^{x} \ln 10}\cdot e^{x}=\frac{1}{\ln 10}=1$$

$$b)\ y=\log_5 (x+1)^2 \Rightarrow \frac{dy}{dx}=\frac{1}{(x+1)^2 \ln 5}\cdot 2(x+1)=\frac{2}{(x+1)\ln 5}$$

$$c)\ y=\log_2 (3x^2+1)^3 \Rightarrow \frac{dy}{dx}=\frac{1}{(3x^2+1)^3 \ln 2}\cdot 3(3x^2+1)^2\cdot 6x=\frac{18x}{(3x^2+1)\ln 2}$$

$$d)\ y=[\ln(x^2+2)^2]^3 \Rightarrow \frac{dy}{dx}=3[\ln (x^2+2)^2]^2\cdot\frac{1}{(x^2+2)^2}\cdot 2(x^2+2)\cdot 2x$$

$$e)\ y+\ln(xy)=1 \Rightarrow \frac{dy}{dx}+\frac{1}{xy}\cdot\left(y+x\frac{dy}{dx}\right)=0 \Rightarrow \frac{dy}{dx}\left(1+\frac{1}{y}\right)=-\frac{1}{x}$$

$$\frac{dy}{dx}=\frac{-\frac{1}{x}}{1+\frac{1}{y}}=-\frac{y}{x\,(y+1)}$$

$$f)\ y=\frac{(2x^3-4)^{\frac{2}{3}}\cdot(2x^2+3)^{\frac{5}{2}}}{(7x^3+4x-3)^2} \Rightarrow \frac{dy}{dx}=$$

$$\frac{(7x^3+4x-3)^2\left[\frac{2}{3}(2x^3-4)^{-\frac{1}{3}}\cdot 6x^2\cdot(2x^2+3)^{\frac{5}{2}}+\frac{5}{2}(2x^2+3)^{\frac{3}{2}}\cdot 4x\cdot(2x^3-4)^{\frac{2}{3}}\right]-\left[(2x^3-4)^{\frac{2}{3}}\cdot(2x^2+3)^{\frac{5}{2}}\right]\cdot 2(7x^3+4x-3)(21x^2+4)}{(7x^3+4x-3)^4}$$

***solve f) in another way*:**

$$\ln y=\frac{2}{3}\ln(2x^3-4)+\frac{5}{2}\ln(2x^2+3)-2\ln(7x^3+4x-3)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{2}{3}\,\frac{6x^2}{(2x^3-4)}+\frac{5}{2}\,\frac{4x}{(2x^2+3)}-2\,\frac{21x^2+4}{(7x^3+4x-3)}$$

$$\frac{dy}{dx}=2y[\frac{2x^2}{(2x^3-4)}+\frac{5x}{(2x^2+3)}-\frac{21x^2+4}{(7x^3+4x-3)}$$

Trigonometric functions: If u is any differentiable function of x, then:

9) $\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$ 10) $\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$

11) $\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$ 12) $\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$

13) $\frac{d}{dx}\sec u = \sec u \cdot \tan u \cdot \frac{du}{dx}$ 14) $\frac{d}{dx}\csc u = -\csc u \cdot \cot u \cdot \frac{du}{dx}$

$Ex\ 9: Find\ \frac{dy}{dx}\ for\ the\ following\ functions:$

$a)\ y = \tan(3x^2)$ $b)\ y = (\csc x + \cot x)^2$ $c)\ y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$

$d)\ y = \tan^2(\cos x)$ $e)\ x + \tan(xy) = 0$ $f)\ y = \sec^4 x - \tan^4 x$

$Sol:$

$$a)\ \frac{dy}{dx} = \sec^2(3x^2)\cdot 6x = 6x\cdot\sec^2(3x^2)$$

$$b)\ \frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x\cot x - \csc^2 x) = -2\csc x\cdot(\csc x + \cot x)^2$$

$$c)\ \frac{dy}{dx} = 2\cos\frac{x}{2}\left(\frac{1}{2}\right) - \left[x\left(-\sin\frac{x}{2}\left(\frac{1}{2}\right)\right) + \cos\frac{x}{2}\right] = \frac{x}{2}\cdot\sin\frac{x}{2}$$

$$d)\ \frac{dy}{dx} = 2\cdot\tan(\cos x)\cdot\sec^2(\cos x)\cdot(-\sin x)$$

$$= -2\cdot\sin x\cdot\tan(\cos x)\cdot\sec^2(\cos x)$$

$$\boldsymbol{e)\ 1 + sec^2(xy)\cdot\left(x\frac{dy}{dx} + y\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1 + y\cdot sec^2(xy)}{x\cdot sec^2(xy)} = -\frac{cos^2(xy) + y}{x}}$$

$$\boldsymbol{f)\ \frac{dy}{dx} = 4sec^3 x\cdot sec\, x\cdot tan\, x - 4\cdot tan^3 x\cdot sec^2 x = 4tan\, x\cdot sec^2 x}$$

$\boldsymbol{Ex\ 10: Prove\ that:}$

$$\boldsymbol{a)\ \frac{d}{dx}tan\, u = sec^2 u\cdot\frac{du}{dx}} \qquad \boldsymbol{b)\ \frac{d}{dx}sec\, u = sec\, u\cdot tan\, u\cdot\frac{du}{dx}}$$

Proof:

$$a)\ L.H.S. = \frac{d}{dx}\tan u = \frac{d}{dx}\cdot\frac{\sin u}{\cos u} = \frac{\cos u\cdot\cos u\cdot\frac{du}{dx} - \sin u\cdot(-\sin u)\frac{du}{dx}}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\cdot\frac{du}{dx} = \frac{1}{\cos^2 u}\cdot\frac{du}{dx} = \sec^2 u\cdot\frac{du}{dx} = R.H.S.$$

$$b)\ L.H.S. = \frac{d}{dx}\sec u = \frac{d}{dx}\frac{1}{\cos u} = -\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$

$$= \frac{1}{\cos u}\cdot\frac{\sin u}{\cos u}\cdot\frac{du}{dx} = \sec u\cdot\tan u\cdot\frac{du}{dx} = R.H.S.$$

Inverse trigonometric functions: If u is any differentiable function of x,then:

15) $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$

16) $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$

17) $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

18) $\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$

19) $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\ \sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$

20) $\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\ \sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$

Ex 11: Find $\frac{dy}{dx}$ *in each of the following functions:*

$a)\ y = \cot^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$ $\qquad b)\ y = \sin^{-1}\frac{x-1}{x+1}$

$c)\ y = x\cdot\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$ $\qquad d)\ y = \sec^{-1}5x$

$e)\ y = x\cdot\ln(\sec^{-1}x)$ $\qquad f)\ y = 3^{\sin^{-1}2x}$

Sol:

$$a)\frac{dy}{dx} = -\frac{1}{1+\frac{4}{x^2}}\cdot\left(-\frac{2}{x^2}\right) + \frac{1}{1+\frac{x^2}{4}}\cdot\left(\frac{1}{2}\right) = \frac{4}{4+x^2}$$

$$b)\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2} = \frac{1}{(x+1)\sqrt{x}}$$

$$c)\frac{dy}{dx} = x\,\frac{-2}{\sqrt{1-4x^2}} + \cos^{-1}2x - \frac{1}{4}\cdot\frac{-8x}{\sqrt{1-4x^2}} = \cos^{-1}2x$$

$$d)\frac{dy}{dx} = \frac{5}{|5x|\sqrt{25x^2-1}} = \frac{1}{|x|\sqrt{25x^2-1}}$$

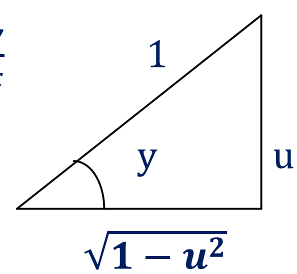
Ex 12: *Prove that*:

$$a)\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad b)\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

Proof: *a*)

$$Let\ y = \sin^{-1}u \Rightarrow u = \sin y \Rightarrow \frac{du}{dx} = \cos y\frac{dy}{dx} = \sqrt{1-u^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \Rightarrow \frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$



Hyperbolic functions: If u is any differentiable function of x, then:

21) $\frac{d}{dx}\sinh u = \cosh u\cdot\frac{du}{dx}$ 22) $\frac{d}{dx}\cosh u = \sinh u\cdot\frac{du}{dx}$

23) $\frac{d}{dx}\tanh u = \text{sech}^2 u\cdot\frac{du}{dx}$ 24) $\frac{d}{dx}\coth u = -\text{csch}^2 u\cdot\frac{du}{dx}$

25) $\frac{d}{dx}\text{sech}\, u = -sech\, u\cdot\tanh u\cdot\frac{du}{dx}$ 26) $\frac{d}{dx}\text{csch}\, u = -csch\, u\cdot\coth u\cdot\frac{du}{dx}$