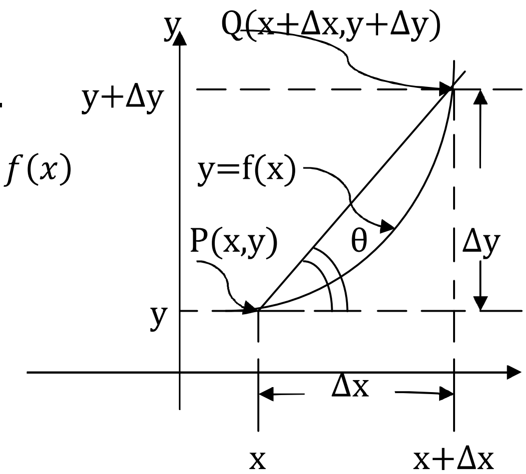
DIFFERENTIATION

If $y = f(x)$ *is a function of* x *defined as a curve on the* $x - y$ *plane.*

Let $P(x, y)$ *is a point on the curve* $y = f(x)$ *, and* $Q(x + \Delta x, y + \Delta y)$ *another point on this curve as see in the figer.*



$y = f(x)$

$y + \Delta y = f(x + \Delta x)$

$y + \Delta y - y = f(x + \Delta x) - f(x)$, *eliminating* y

$\Delta y = f(x + \Delta x) - f(x)$, *divided both sides by* Δx

$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

If the limit of $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ *as* $\Delta x \to 0$ *exists and finite,* $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = \frac{dy}{dx}$

We call this limit the derivative of f *at* x *and say that* f *is differentiable at* x.

$m = slope\ of\ the\ line = \tan\theta = \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$

DEFINITION: Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$,
provided the limit exists.

$Ex1: Find\ the\ derivative\ of\ the\ function: y = \frac{1}{\sqrt{2x+3}}$

$Sol:$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{2(x+\Delta x)+3}} - \frac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x \cdot \sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} \cdot \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x \cdot \sqrt{2(x+\Delta x)+3}\sqrt{2x+3}\left(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}\right)}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

Rules of derivatives: **Let c and n are constants, u , v and w are differentiable functions of x:**

1. $\frac{d}{dx} c = 0$
2. $\frac{d}{dx} u^n = n\, u^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$
3. $\frac{d}{dx}\, c\, u = c \frac{du}{dx}$
4. $\frac{d}{dx}(u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$; $\frac{d}{dx}(u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$
5. $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$, and
   $\frac{d}{dx}(u \cdot v \cdot w) = u \cdot v \frac{dw}{dx} + u \cdot w \frac{dv}{dx} + v \cdot w\, du/dx$
6. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $\quad where\ v \neq 0$

$Ex2: find \frac{dy}{dx}\ for\ the\ following\ functions:$

$a)\ y = (x^2+1)^5$ $\qquad b)\ y = [(5-x)(4-2x)]^2$

$c)\ y = (2x^3 - 3x^2 + 6x)^{-5}$ $\qquad d)\ y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

$e)\ y=\frac{(x^2+x)(x^2-x+1)}{x^3}$ $\qquad f)\ y=\frac{x^2-1}{x^2+x-2}$

Sol:

$a)\frac{dy}{dx}=5(x^2+1)^4\cdot 2x=10x(x^2+1)^4$

$b)\frac{dy}{dx}=2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$=8(5-x)(2-x)(2x-7)$

$c)\frac{dy}{dx}=-5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$=-30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

$d)\ y=12x^{-1}-4x^{-3}+3x^{-4}\Rightarrow\frac{dy}{dx}=-12x^{-2}+12x^{-4}-12x^{-5}$

$\Rightarrow\frac{dy}{dx}=-\frac{12}{x^2}+\frac{12}{x^4}-\frac{12}{x^5}$

$$e)\frac{dy}{dx}=\frac{x^3[(x^2-x+1)+(x+1)(2x-1)]-3x^2(x+1)(x^2-x+1)}{x^6}$$

$=-\frac{3}{x^4}$

$$f)\frac{dy}{dx}=\frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2}=\frac{x^2-2x+1}{(x^2+x-2)^2}$$

The Chain Rule:

1. *If y is a differentiable function of t and x is a differentiable function of t, then* $\frac{dy}{dx}$ *will be*:

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

2. *If y is a differentiable function of t and t is a differentiable function of x, then* $\frac{dy}{dx}$ *will be*:

$$\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

***Ex3**: Use the chain rule to express* $\frac{dy}{dx}$ *in terms of x and y*

a) $y = \frac{t^2}{t^2+1}$ and $t = \sqrt{2x+1}$

b) $y = \frac{1}{t^2+1}$ and $x = \sqrt{4t+1}$

c) $y = \left(\frac{t-1}{t+1}\right)^2$ and $x = \frac{1}{t^2} - 1$ at $t = 2$

d) $y = 1 - \frac{1}{t}$ and $t = \frac{1}{1-x}$ at $x = 2$

Sol:

a) $y = \frac{t^2}{t^2+1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2+1)-2t.t^2}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$

$t = (2x+1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2} \cdot (2x+1)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2t}{(t^2+1)^2} \cdot \frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{\left((2x+1)+1\right)^2} \cdot \frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$

b) $y = (t^2+1)^{-1} \Rightarrow \frac{dy}{dt} = -2t(t^2+1)^{-2} = -\frac{2t}{(t^2+1)^2}$

$x = (4t+1)^{\frac{1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}(4t+1)^{-\frac{1}{2}} \cdot 4 = \frac{2}{\sqrt{4t+1}}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{2t}{(t^2+1)^2} \div \frac{2}{\sqrt{4t+1}} = -\frac{t\sqrt{4t+1}}{(t^2+1)^2}$

$= -\frac{x^2-1}{4} \cdot x \cdot \frac{1}{y^2} = -\frac{x(x^2-1)}{4y^2}$

$$\text{where } x = \sqrt{4t+1} \Rightarrow t = \frac{x^2-1}{4}$$

$$where\ \ y = \frac{1}{t^2+1} \Rightarrow t^2+1 = \frac{1}{y}$$

Higher derivatives:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x) = \lim_{\Delta x \to 0}\frac{f'(x+\Delta x) - f'(x)}{\Delta x}$$

if the limit exists.

This derivative is called the second derivative of y with respect to x.

It is written in a number of ways, for example, y'', $f''(x)$ *or* $\frac{d^2f(x)}{dx^2}$

In the same manner we may define third and higher derivatives,

using similar notations. The nth derivative may be written:

$$y^{(n)}, \qquad f^{(n)}(x), \qquad \frac{d^ny}{dx^n}.$$

Ex4: Find all derivatives of the following function:

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol:

$$\frac{dy}{dx} = 9x^2 - 8x + 7, \qquad \frac{d^2y}{dx^2} = 18x - 8, \qquad \frac{d^3y}{dx^3} = 18$$

$$\frac{d^4y}{dx^4} = 0 = \frac{d^5y}{dx^5} = \cdots\cdots$$

Ex 5: Find the third derivative of the following function:

$$y = \frac{1}{x} + \sqrt{x^3}$$

Sol: $\frac{dy}{dx} = -\frac{1}{x^2} + \frac{3}{2}x^{\frac{1}{2}}, \qquad \frac{d^2y}{dx^2} = \frac{2}{x^3} + \frac{3}{4}x^{-\frac{1}{2}}$

$$\frac{d^3y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8}x^{-\frac{3}{2}} \quad \Rightarrow \frac{d^3y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

Implicit Differentiation:

If the formula for f is an algebraic combination of powers of x and y.

To calculate the derivatives of these implicity defined functions, we simply differentiate both sides of the defining equation with respect to x.

*Ex*6: *Find* $\frac{dy}{dx}$ *for the following functions*:

a) $x^2 \cdot y^2 = x^2 + y^2$ b) $(x+y)^3 + (x-y)^3 = x^4 + y^4$

c) $\frac{x-y}{x-2y} = 2$ at $P(3,1)$ d) $xy + 2x - 5y = 2$ at $P(3,2)$

Sol:

$$a)\ x^2\left(2y\frac{dy}{dx}\right) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2y - y}$$

$$b)\ 3(x+y)^2\left(1+\frac{dy}{dx}\right) + 3(x-y)^2\left(1-\frac{dy}{dx}\right) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c)\ \frac{(x-2y)\left(1-\frac{dy}{dx}\right) - (x-y)\left(1-\frac{2dy}{dx}\right)}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}_{(3,1)} = \frac{1}{3}$$

$$d)\ x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \left[\frac{y+2}{5-x}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

Exponential functions: If u is any differentiable function of x, then:

$$7)\ \frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx} \qquad and \quad \frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$