Root Locus Techniques

The root locus is a powerful technique as it brings into focus the complete dynamic response of the system and further, being a graphical technique an approximate root locus sketch can be made quickly and the designer can easily visualize the effects of varying system parameters on root locations.

Feedback control systems are difficult to comprehend from a qualitative point of view, and hence they rely heavily upon mathematics. So gains and other parameters were designed to yield a desired transient response for only first and second order systems. The root locus can be used to solve the same kind of problem, its real power lies in its stability to provide solution for systems of order higher than two.

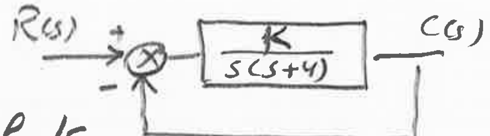
The root locus can be used to describe qualitatively the performance of the system as various parameters are changed such that the effect of varying gain (K), upon percent overshoot, settling time, and peak time.

The root locus also gives a graphic representation of a system stability we can clearly see ranges of stability or instability.

The root locus Concepts

Consider the control system represented by figure

the characteristic equation is



$$s^2+4s+K=0. \qquad —(1)$$

the roots of ch-eq. depend upon the value of K

the roots of eq (1) are $s_{1,2}=a+jb$

$$a=-2, \quad b=\sqrt{K-a^2}=\sqrt{K-4}$$

For $K>4 \Rightarrow r_{1,2}=-2\pm j\sqrt{K-4}$

$K=4 \Rightarrow r_{1,2}=-2$

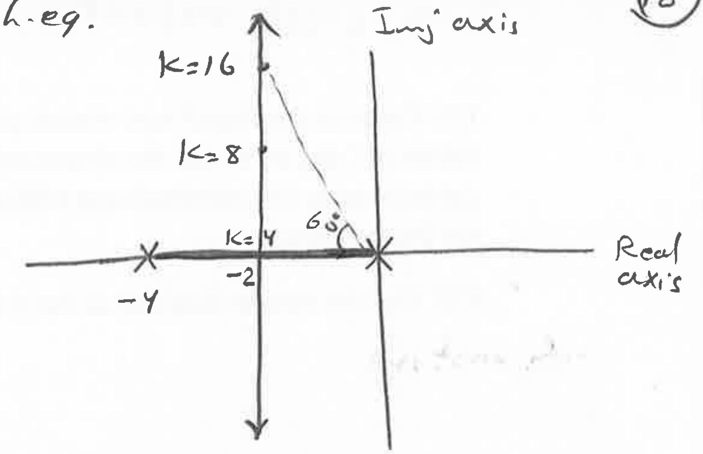
$K<4 \Rightarrow r_{1,2}=-2\mp\sqrt{4-K}$

this figure is a plot of the roots of Ch.eq. for various value of k,

when $k = 0$, $S_{1,2} = 0, -4$

$k = 4$, $S_{1,2} = -2$

$k = 16$, $S_{1,2} = -2 \pm j\sqrt{12}$



sketching a Root Locus

- The root locus approaches straight lines as asymptotes, the equation of asymptotes is given by the real axis intercept σ_a , and angle θ_a as

$$\sigma_a = \frac{\sum \text{poles} - \sum \text{zeros}}{\text{n. of Poles} - \text{n of zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{n_{poles} - n_{zeros}}$$

- Real axis breakaway and breakin points

root loci appear to break away from the real axis as a system poles move from real axis to the complex plane.
to find the points at which the root locus breaks away from breaks in to the real axis by maximize and minimize the gain.

where the ch-eq. yield $k = -\dfrac{1}{G(s)\,H(s)}$

then $\dfrac{dk(s)}{ds} = 0 \Rightarrow$ to get the value of S which are the breakaway and break in points.

- Frequancy and Gain at Imaginary - axis Crossing

the frequancy ω and gain k for which the root locus crosses the imaginary axis can be find by using Routh's Array to find the value of k and then change $(s \rightarrow \omega j)$ in ch.eq to find the frequancy (ω) in rad/sec

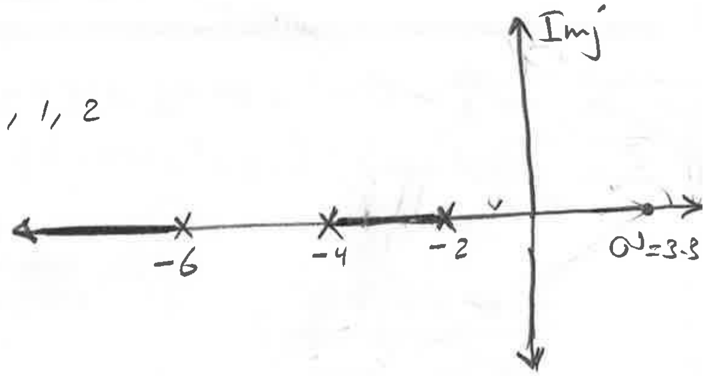
Ex:1 sketch the root locus for a unity feedback system that has forward transfer function $G(s) = \dfrac{K}{(s+2)(s+4)(s+6)}$

$$\sigma_a = \frac{\sum P - \sum Z}{P - Z} = \frac{-12 - 0}{3 - 1} = \frac{-12}{3} = -4$$

$$\theta_a = \frac{(2k+1)\pi}{P-Z} = \frac{\pi}{3} \quad , \quad k = 0, 1, 2$$

$$= \frac{3\pi}{3} = \pi$$

$$= \frac{5\pi}{3}$$



the G(s), has three poles only $s = -2, -4, -6$
the area of roots take from first (pole or zero) at right hand in real axis. (the dark area)

ch.eq of system is $(s+2)(s+4)(s+6) + k = 0$

$$K(s) = -(s^3 + 12s^2 + 44s + 48$$

$$\frac{dk}{ds} = -(3s^2 + 24s + 44) = 0$$

$$s^2 + 8s + 14.67 = 0$$

$$s_{1,2} = \frac{-8 \pm \sqrt{64 - 58.86}}{2}$$

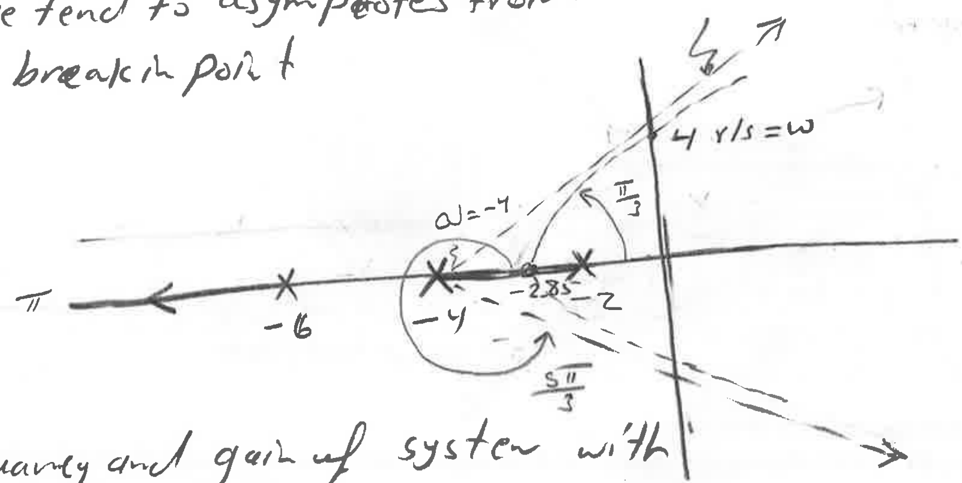
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -5.15 \quad , \quad -2.85$$

خارج منطقة الجذور تهمل

تأخذ داخل المنطقة

the poles will move tend to asymptotes from breakaway and breakin point



to find the frequancy and gain of system with imaginary axis by

Ch-eq $\Longrightarrow s^3 + 12s^2 + 44s + (48 + K) = 0$

s^3	1	44
s^2	12	$48+K$
s^1	$\frac{528-(48+K)}{12}$	
s^0	$48+K$	

, to stay system stable must be $\frac{528-(48+K)}{12} > 0$

$528 - 48 - K > 0$

$K = 480$

For $K = 480$, the roots cross the imaginery axis. and to find frequancy ~~by~~ butting $s = \omega j$ in Ch-eq.

$(\omega j)^3 + 12(\omega j)^2 + 44\omega j + 48 + K = 0$

$-\omega^3 j - 12\omega^2 + 44\omega j + 48 + K = 0$

$(44\omega - \omega^3)j + (48 + K - 12\omega^2) = 0$

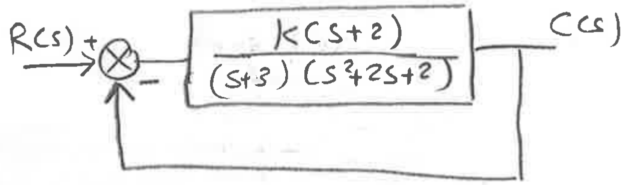
Img Real

$44\omega - \omega^3 = 0 \Longrightarrow \omega^2 = 44, \ \omega_{1,2} = \pm 6.63$

and $K \Longrightarrow 48 + K - 12(44) = 0$

$K = 480$

Ex: 2/ Given the unity feedback system of figure, find the angle of departure from the complex poles and sketch the root locus.



Sol

Poles , $-3, -1+j, -1-j$

Zeros , -2

$$\sigma_a = \frac{\sum P - \sum Z}{P - Z} = \frac{(-3-1-1)-(-2)}{3-1}$$

$$= \frac{-5+2}{2} = -1.5$$

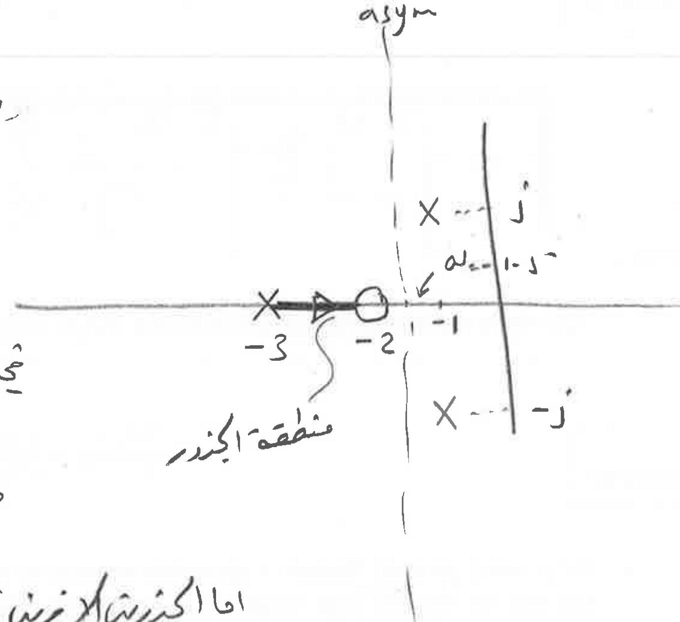
$$s^2+2s+2=0$$

$$s_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= -1 \pm j$$

$$\theta_a = \frac{(2k+1)\pi}{P-Z} \quad , k = 0, 1 \quad P-Z = 2$$

$$\theta_a = \frac{\pi}{2} \, , \, \frac{3\pi}{2}$$



في منطقة الجذر يتحرك دائما poles نحو ← Zeros

X ⟶ O

و يكتفي بهذه الحركة

اما الجذرين الاخرين فيجب ان يتحركا باتجاه المحاذيات وتكون حركتهما

شاطرتين زاوية تسمى زاوية المغادرة

to find the angle of departure, do

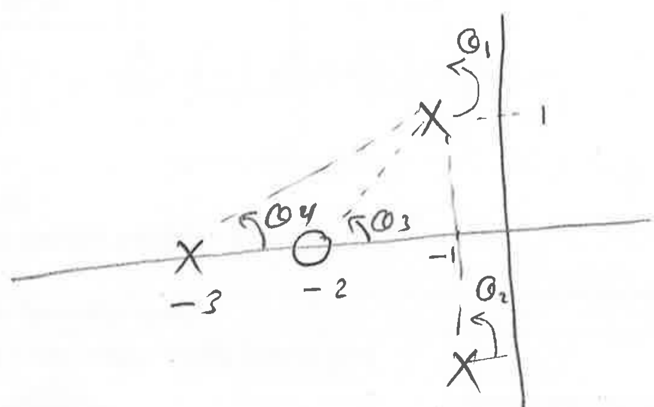
$$\sum \theta_{i\,zeros} - \sum \theta_{i\,poles} = 180$$

$$\theta_3 - \theta_1 - \theta_2 - \theta_4 = 180$$

$$45 - \theta_1 - 90 - 26.5 = 180$$

$$\theta_1 = -251.56^\circ$$

$$= 108.4^\circ$$

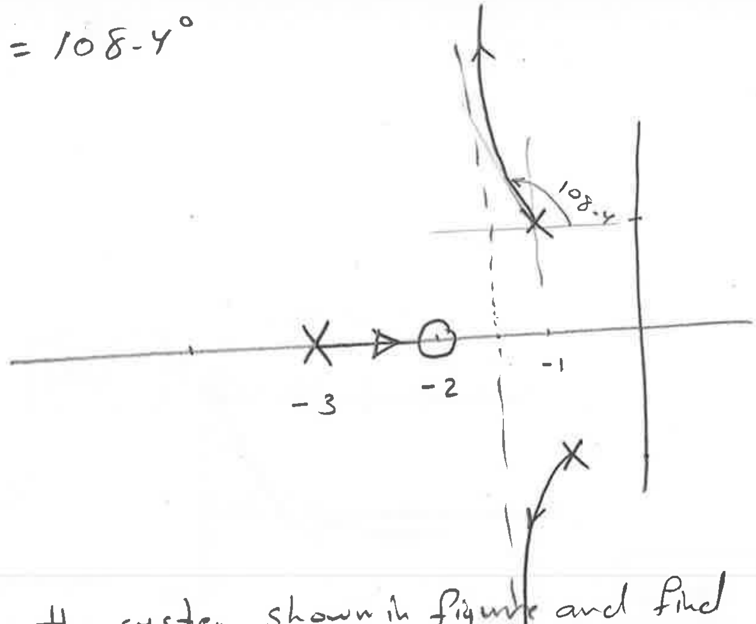


$$\theta_3 = \tan^{-1}\frac{1}{1} = 45^\circ$$

$$\theta_4 = \tan^{-1}\frac{1}{2} = 26.5^\circ$$

$$\theta_2 = 90$$

the angle of departure $\theta_1 = 108.4^\circ$



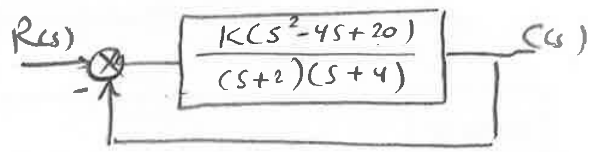
Ex:3 / sketch the root locus for the system shown in figure and find

① The exact point and gain where the locus crosses the 0.45 damping ratio line.

② The exact point and gain where the locus crosses the $j\omega$-axis

③ The breakaway point on the real axis

④ The range of K within which the system is stable.



Sol

Zeros $\Rightarrow$ $s^2 - 4s + 20 = 0$, $s_{1,2} = \dfrac{4 \pm \sqrt{16-80}}{2}$

$= 2 \pm 4j$

poles $\Rightarrow$ $s = -2, -4$

$\sigma_a = \dfrac{\Sigma p - \Sigma z}{p-z} = \dfrac{-6-(-2-2)}{2-2} = \dfrac{-2}{0}$ No. asymptotes

θ_a , No angle.

Ch-eq $\Rightarrow$ to find breakaway $\Rightarrow$ $G(s) = \dfrac{K(s^2-4s+20)}{(s+2)(s+4)}$

$1 + G(s)H(s) = 0 \Rightarrow 1 + \dfrac{K(s^2-4s+20)}{(s+2)(s+4)} = 0$

$k = \dfrac{s^2+6s+8}{s^2-4s+20} \Rightarrow \dfrac{dk}{ds} = \dfrac{(s^2+6s+8)(2s-4) - (s^2-4s+20)(2s+6)}{(s^2-4s+20)^2}$

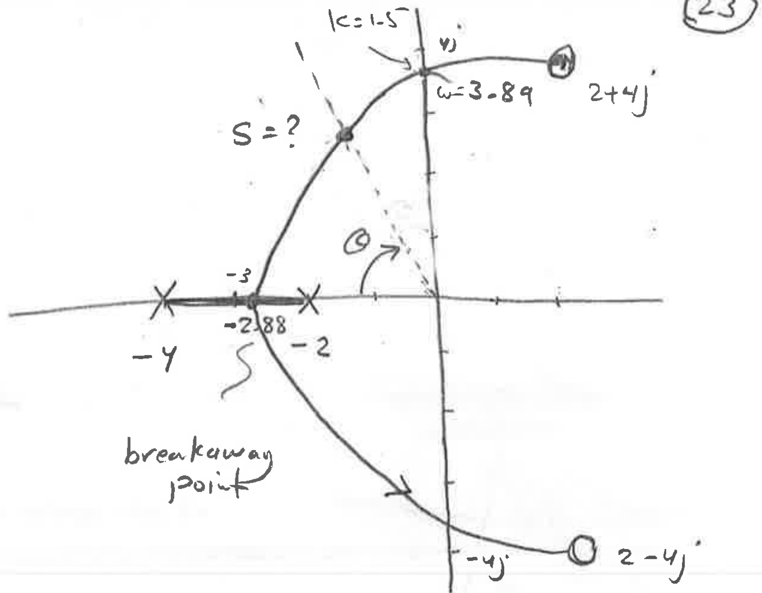
$\therefore \cancel{2s^3} - 4s^2 + 12s^2 - 24s + 16s - 32 - (\cancel{2s^3} + 6s^2 - 8s^2 - 24s + 40s + 120) = 0$

$10s^2 - 24s - 152 = 0$

$s^2 - 2.4s - 15.2 = 0 \Rightarrow s_{1,2} = \dfrac{2.4 \pm \overset{8.15}{\sqrt{2.4^2 + 60.8}}}{2}$

$= 5.28, -2.88$

to find the exact point and gain at crosses with 0.45 damping ratio.



$\therefore \cos^{-1} 0.45 = 63.25^\circ = \theta$

S من الرسم الدقيق ممكن تقدير قيمة

$S = -1.5 \pm 3j$ Sub in Ch.eq.

$$|G(s)\,H(s)| = 1$$

$$\left| \frac{K(s^2 - 4s + 20)}{(s+2)(s+4)} \right|_{s=-1.5+3j} = 1 \quad \text{to find } K \text{ in this point}$$

$$\left| \frac{K\left[(-1.5+3j)^2 - 4(-1.5+3j) + 20\right]}{\underset{(0.5+3j)}{(-1.5+3j+2)}\ \underset{(2.5+3j)}{(-1.5+3j+4)}} \right| = 1$$

$$\left| \frac{K(2.25 - 4.5j - 9 + 6 - 12j + 20)}{1.25 + 1.5j + 7.5j - 9} \right| = 1 \Rightarrow K\, \frac{19.25 - 16.5j}{-7.75 + 9j}$$

$$K\, \frac{\overbrace{\sqrt{19.25^2 + (16.5)^2}}^{25.35}}{\underbrace{\sqrt{(7.75)^2 + (9)^2}}_{11.87}} = 1 \Rightarrow K = \frac{11.87}{25.35} = 0.4$$

to find the exact point and gain with crosses at imaginary axis by Routh or put $s \Rightarrow \omega j$ in ch.eq.

Ch-eq $\Rightarrow (s+2)(s+4) + K(s^2 - 4s + 20) = 0$.

$s^2 + 6s + 8 + Ks^2 - 4Ks + 20K = 0$

$(1+K)s^2 + (6-4K)s + (8+20K) = 0$.

	s^3	$1+k$	$8+20k$
auxiliary ناخذ من معادلة	s^2	$6-4k$	0
او من الصف الذي يكون صفرا	s^1	$8+20k$	

$6 - 4K = 0$

$\therefore K = 1.5$

sub $k = 1.5$ to find frequancy

$$(1+1.5)(\omega j)^2 + (6-4(1.5))\omega j + (8 + 20 * 1.5) = 0$$

$$-2.5\omega^2 + 38 = 0 \Rightarrow \omega^2 = 15.2 \Rightarrow \omega = 3.89 \text{ rad/sec.}$$

Ex: 4 The T.F of a unity feedback control system is given by $G(s) = \dfrac{K}{s(s+2)(s+4)}$

Draw the root locus also determine

i/ the value of K to have 45% overshoot for unit step input

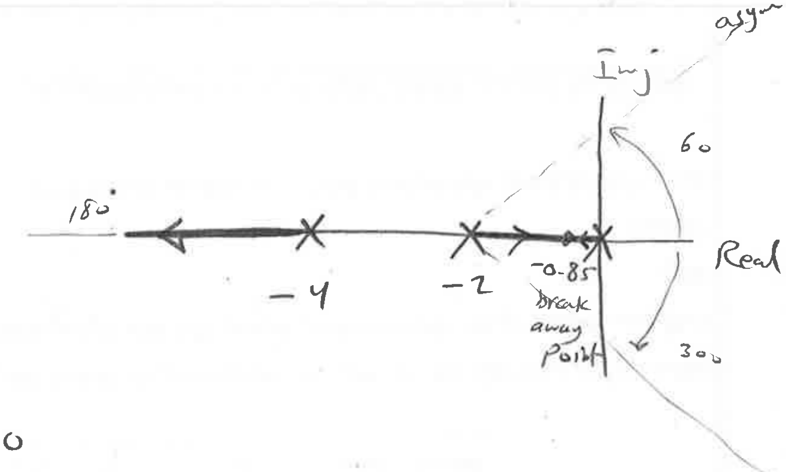
ii/ " " " K for sustained oscillation in output

iii/ " " " K_v corresponding to value of K as obtained in (i)

iv/ the value of settling time ts.

sol

- poles / 0, −2, −4



- $\sigma_a = \dfrac{\Sigma P - \Sigma Z}{P - Z} = \dfrac{-6-0}{3-0} = -2$

- $\theta_a = \dfrac{(2k+1)\pi}{P-Z}$, $k = 0, 1, 2$

$\theta = \pi/3, \pi, \dfrac{5\pi}{3}$

- breakaway point $\rightarrow 1 + G(s)H(s) = 0$

$$\therefore K = -s(s+2)(s+4)$$

$$K = -(s^3 + 6s^2 + 8s)$$

$$\frac{dK}{ds} = -(3s^2 + 12s + 8) = 0$$

$$s_{1,2} = -3.15, -0.85$$

- intersection with the imaginary axis , $1 + G(s)H(s) = 0$.

$$s(s+2)(s+4) + K = 0$$

$$s^3 + 6s^2 + 8s + K = 0.$$

s^3	1	8
s^2	6	K
s^1	$\frac{48-K}{6}$	
s^0	K	

the critical value of $K = 48$

to find frequancy at this value of K

$$6s^2 + K = 0. \Rightarrow s^2 = -\frac{48}{6}$$

$$\therefore s_{1,2} = \pm j\sqrt{8} = \pm 2.8j$$

The max. overshoot is given $Mp\% = e^{\frac{-\pi \xi}{\sqrt{1-\xi^2}}} \times 100$

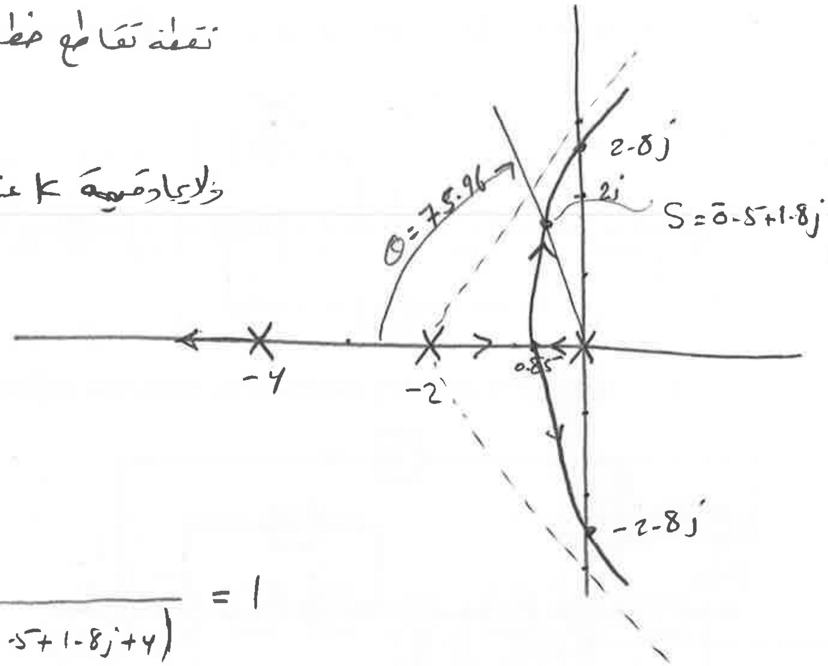
$$Mp = 45\%$$

$$\frac{45}{100} = e^{\frac{-\pi \xi}{\sqrt{1-\xi^2}}} \Longrightarrow \xi = 0.24$$

$$\therefore \cos \theta = \xi \Longrightarrow \theta = 75.96^\circ$$

نقطة تقاطع خط الزاوية θ مع رسم منحنى الجذور يكون في نقطة

$$s = -0.5 \pm 1.8j$$



ولإيجاد قيمة K عند تلك النقطة نعوض قيمة S في معادلة

$$|G(s)H(s)| = 1$$

$$\left| \frac{K}{s(s+2)(s+4)} \right| = 1$$

$$\frac{K}{|-0.5+1.8j|\,|-0.5+1.8j+2|\,|-0.5+1.8j+4|} = 1$$

$$\frac{K}{|2.34|\,|1.5+1.8j|\,|3.5+1.8j|} = 1 \Longrightarrow \frac{K}{1.86 * 2.34 * 3.93} = 1 \Longrightarrow K = 17.13$$

$$-K_v = \lim_{s \to 0} s \cdot G(s)H(s) = \lim_{s \to 0} \cancel{s} \cdot \frac{K = 17.13}{\cancel{s}(s+2)(s+4)} = \frac{17.13}{2*4} = 2.14$$

$$\therefore S = -0.5 + 1.8j$$
$$= -\xi \omega_n + \omega_d j \qquad \text{by comparing}$$

$$\therefore \xi \omega_n = 0.5$$

$$\therefore t_s = \frac{4}{\xi \omega_n} = \frac{4}{0.5} = 8 \text{ sec.}$$

Ex:5 Draw the root locus plot for the following control syst

$$G(s)H(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

sol

Poles, $0, -1$

Zeros, $-2, -3$

$\sigma_a = \frac{-1-(-5)}{2-2} = \frac{4}{0}$ No. asymptote, and No, angle θ_a

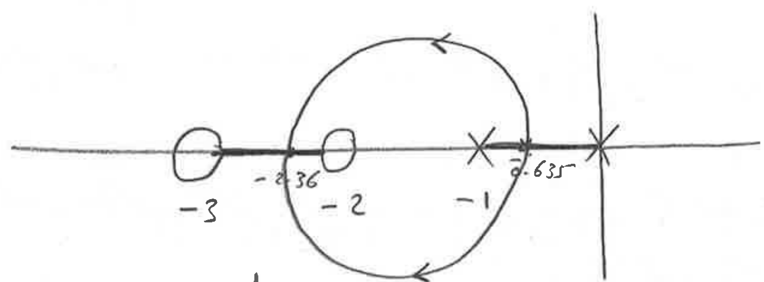
breakaway $\Rightarrow$ $1+G(s)H(s)=0 \Rightarrow K = \frac{-s(s+1)}{(s+2)(s+3)} = \frac{-(s^2+s)}{(s^2+5s+6)}$

$$\frac{dK}{ds} = -\frac{[(s^2+5s+6)(2s+1)-(s^2+s)(2s+5)]}{(s^2+5s+6)^2} = -\frac{[4s^2+12s+6]}{(s^2+5s+6)^2} = 0$$

$\Rightarrow 4s^2+12s+6=0 \Rightarrow 2s^2+6s+3=0 \Rightarrow s_{1,2} = \frac{-6 \pm \sqrt{36-4\times2\times3}}{2\times2}$

نلاحظ ان هذه القيم تكون ضمن منطقة الجذور $= -2.36, -0.635$

ولذلك سوف تتحرك X نحو O من هذه النقاط



H-W Draw the root locus for the following systems

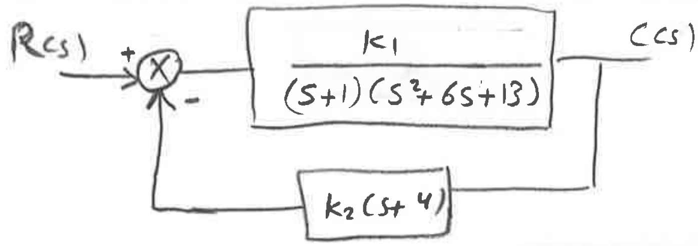
① $G(s)H(s) = \frac{K}{(s^2+2s+2)(s^2+2s+5)}$

② $G(s)H(s) = \frac{K(s+2)}{s(s^2+3s+4)}$

③ $G(s)H(s) = \frac{K(s^2+s+4)}{s(s+4)(s+10)}$, find the value of K at $\zeta = 0.5$

④ $G(s)H(s) = \frac{K(s+2)}{s(s+1)(s^2+4s+8)}$

5



6 sketch the root locus for each of the following characteristic equations

1. $s(s^2+8s+25) + K(s+2) = 0$
2. $s(s^2+8s+25) + K(s+2)(s+4) = 0$.