Al-Mustaqbal university

Engineering technical college

Department of Building
&Construction Engineering

Mathematics

First class

Lecture No.8

Assist. Lecture

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2. Inverse trigonometric functions

Inverse trigonometric functions are simply defined as the inverse functions of the basic trigonometric functions which are sine, cosine, tangent, cotangent, secant, and cosecant functions. They are also termed as arcus functions, antitrigonometric functions or cyclometric functions. These inverse functions in trigonometry are used to get the angle with any of the trigonometry ratios. The inverse trigonometry functions have major applications in the field of engineering, physics, geometry and navigation.

What are Inverse Trigonometric Functions?

Inverse trigonometric functions are also called “**Arc Functions**” since, for a given value of trigonometric functions, they produce the length of arc needed to obtain that particular value. The inverse trigonometric functions actually performs the opposite operation of the trigonometric functions such as sine, cosine, tangent, cosecant, secant, and cotangent. We know that, trig functions are specially applicable to the right angle triangle. These six important functions are used to find the angle measure in a right triangle when two sides of the triangle measures are known.

Formulas

The basic inverse trigonometric formulas are as follows:

Inverse Trig Functions	**Formulas**
Arcsine	$\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1, 1]$
Arccosine	$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, $x \in [-1, 1]$
Arctangent	$\tan^{-1}(-x) = -\tan^{-1}(x)$, $x \in R$
Arccotangent	$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$, $x \in R$
Arcsecant	$\sec^{-1}(-x) = \pi - \sec^{-1}(x)$, $\lvert x \rvert \geq 1$
Arccosecant	$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$, $\lvert x \rvert \geq 1$

Inverse Trigonometric Functions Graphs

There are particularly six inverse trig functions for each trigonometry ratio. The inverse of six important trigonometric functions are:

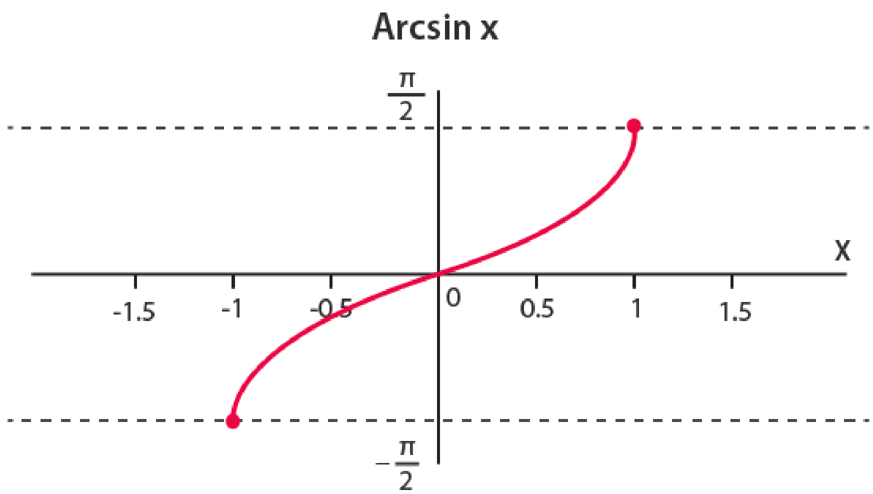
- Arcsine
- Arccosine
- Arctangent
- Arccotangent
- Arcsecant

- Arccosecant

Let us discuss all the six important types of inverse trigonometric functions along with its definition, formulas, graphs, properties and solved examples.

Arcsine Function

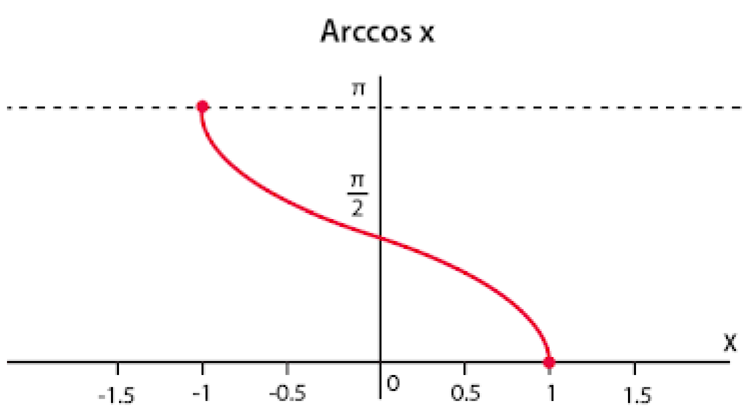
Arcsine function is an inverse of the sine function denoted by $\sin^{-1}x$. It is represented in the graph as shown below:



Domain	$-1 \le x \le 1$
Range	$-\pi/2 \le y \le \pi/2$

Arccosine Function

Arccosine function is the inverse of the cosine function denoted by $\cos^{-1}x$. It is represented in the graph as shown below:

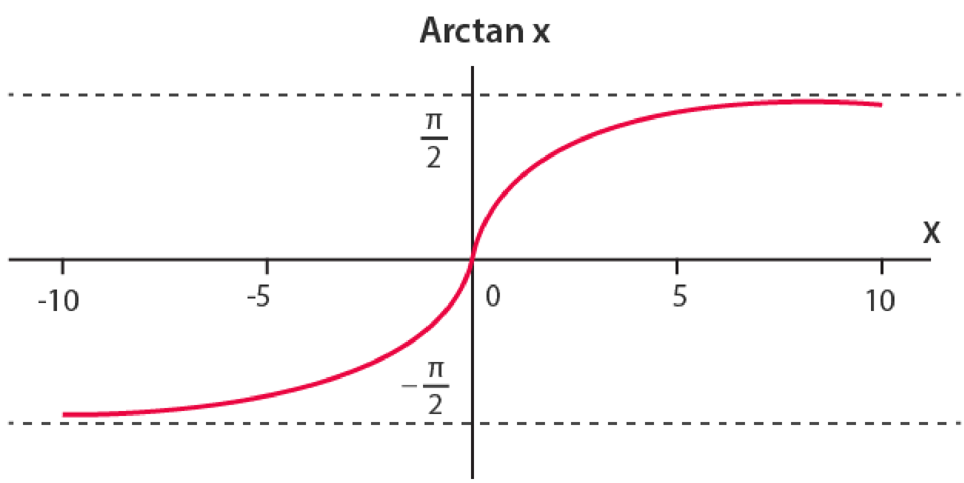


Therefore, the inverse of cos function can be expressed as; **y = cos^{-1}x (arccosine *x*)Domain & Range of arcsine function:**

Domain	$-1 \le x \le 1$
Range	$0 \le y \le \pi$

Arctangent Function

Arctangent function is the inverse of the tangent function denoted by tan^{-1}*x*. It is represented in the graph as shown below:

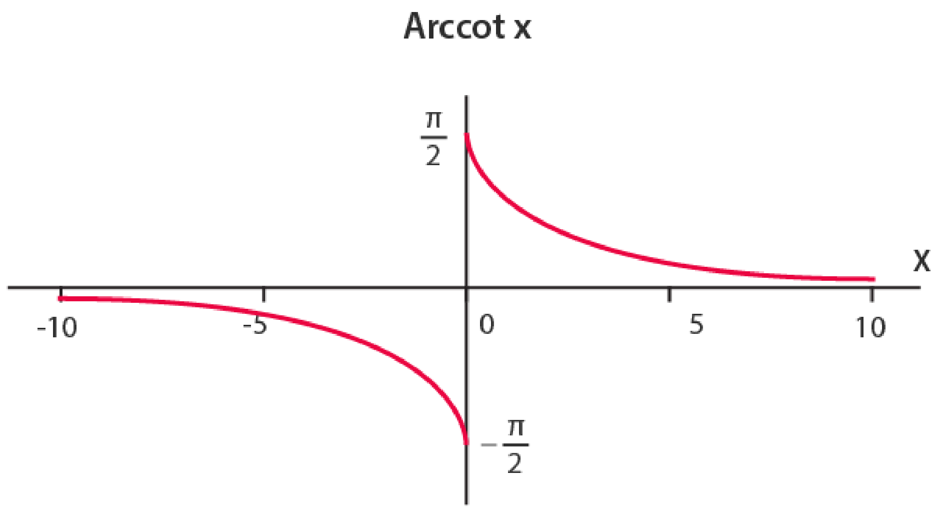


Therefore, the inverse of tangent function can be expressed as; **y = tan^{-1}x (arctangent *x*)Domain & Range of Arctangent:**

Domain	$-\infty < x < \infty$
Range	$-\pi/2 < y < \pi/2$

Arccotangent (Arccot) Function

Arccotangent function is the inverse of the cotangent function denoted by cot^{-1}*x*. It is represented in the graph as shown below:

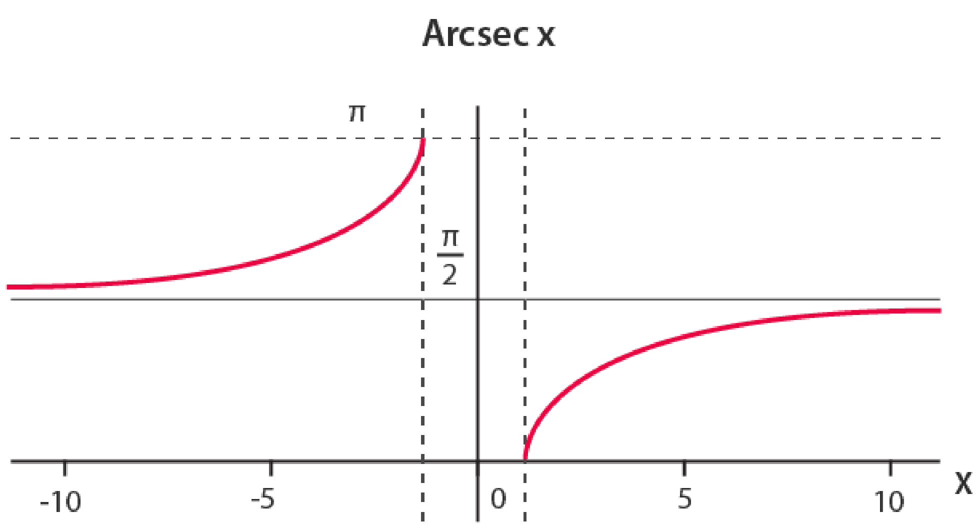


Therefore, the inverse of cotangent function can be expressed as; **y = cot^{-1}x (arccotangent *x*)** Domain & Range of Arccotangent:

Domain	-∞ < x < ∞
Range	0 < y < π

Arcsecant Function

What is arcsecant (arcsec)function? Arcsecant function is the inverse of the secant function denoted by sec^{-1}*x*. It is represented in the graph as shown below:

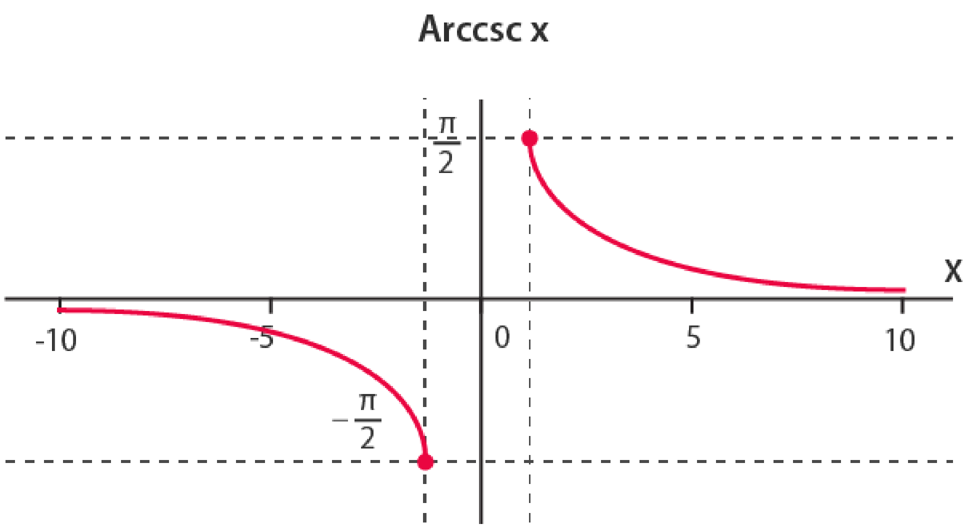


Therefore, the inverse of secant function can be expressed as; **y = sec^{-1}x (arcsecant *x*)Domain & Range of Arcsecant:**

Domain	-∞ ≤ x ≤ -1 or 1 ≤ x ≤ ∞
Range	-π/2 < y < π/2 ; y≠0

Arccosecant Function

What is arccosecant (arccsc *x*) function? Arccosecant function is the inverse of the cosecant function denoted by cosec^{-1}*x*. It is represented in the graph as shown below:



Therefore, the inverse of cosecant function can be expressed as; **y = cosec^{-1}x (arccosecant *x*)Domain & Range of Arccosecant**

Domain	-∞ ≤ x ≤ -1 or 1 ≤ x ≤ ∞
Range	-π/2 < y < π/2 ; y≠0

Inverse Trigonometric Functions Table

Let us rewrite here all the inverse trigonometric functions with their notation, definition, domain and range.

Function Name	Notation	Definition	Domain of x	Range
Arcsine or inverse sine	y = sin^{-1}(x)	x=sin y	−1 ≤ x ≤ 1	• − π/2 ≤ y ≤ π/2 • -90°≤ y ≤ 90°

Arccosine or inverse cosine	$y=\cos^{-1}(x)$	x=cos y	$-1 \le x \le 1$	• $0 \le y \le \pi$ • $0° \le y \le 180°$
Arctangent or Inverse tangent	$y=\tan^{-1}(x)$	x=tan y	For all real numbers	• $-\pi/2 < y < \pi/2$ • $-90° < y < 90°$
Arccotangent or Inverse Cot	$y=\cot^{-1}(x)$	x=cot y	For all real numbers	• $0 < y < \pi$ • $0° < y < 180°$
Arcsecant or Inverse Secant	$y = \sec^{-1}(x)$	x=sec y	$x \le -1$ or $1 \le x$	• $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$ • $0° \le y < 90°$ or $90° < y \le 180°$
Arccosecant	$y=\csc^{-1}(x)$	x=csc y	$x \le -1$ or $1 \le x$	• $-\pi/2 \le y < 0$ or $0 < y \le \pi/2$ • $-90° \le y < 0°$ or $0° < y \le 90°$

Inverse Trigonometric Functions Derivatives

The derivatives of inverse trigonometric functions are first-order derivatives. Let us check here the derivatives of all the six inverse functions.

Inverse Trig Function	**dy/dx**
$\sin^{-1}(x)$	$1/\sqrt{(1-x^2)}$
$\cos^{-1}(x)$	$-1/\sqrt{(1-x^2)}$
$\tan^{-1}(x)$	$1/(1+x^2)$
$\cot^{-1}(x)$	$-1/(1+x^2)$
$\sec^{-1}(x)$	$1/[\lvert x\rvert\sqrt{(x^2-1)}]$
$\csc^{-1}(x)$	$-1/[\lvert x\rvert\sqrt{(x^2-1)}]$

Inverse Trigonometric Functions Properties

The inverse trigonometric functions are also known as Arc functions. Inverse Trigonometric Functions are defined in a certain interval (under restricted domains).

Trigonometry Basics

Trigonometry basics include the basic trigonometry and trigonometric ratios such as sin x, cos x, tan x, cosec x, sec x and cot x. The following article from BYJU'S discusses the basic definition of another tool of trigonometry – Inverse Trigonometric Functions.

Inverse Trigonometric Functions Problems

Example 1: Find the value of x, for sin(x) = 2.

Solution: Given: sinx = 2 x =$\sin^{-1}$(2), which is not possible.

Hence, there is no value of x for which sin x = 2; since the domain of $\sin^{-1}$x is -1 to 1 for the values of x.

Example 2: Find the value of $\sin^{-1}$(sin (π/6)).

Solution:

$\sin^{-1}$(sin (π/6) = π/6 (Using identity $\sin^{-1}$(sin (x)) = x)

Example 1:

$$\sin^{-1}\left(sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Example 2:

$$\sin^{-1}\left(sin\frac{2\pi}{3}\right)$$

$$\pi - \theta$$

$$\pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

$$\sin^{-1}\left(sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Example 3:

$$\cos\left(\cos^{-1}(3)\right) = undefined$$

Example 4:

$$\sin^{-1}(\cos 60)$$

$$\sin^{-1} = \frac{1}{2} = 30 = \frac{\pi}{6}$$

Example 5:

$$\tan(\tan^{-1}\frac{2\pi}{6}) = \frac{2\pi}{6}$$

Example 6:

$$\cos^{-1}(cos\frac{5\pi}{4}))$$

$$2\pi - \theta$$

$$2\pi - \frac{5\pi}{4} = \frac{3\pi}{4}$$

Example 7:

$$f(x) = \sin x + 5 \; find\ the\ domain\ and\ range\ for\ f(x)^{-1}$$

$$y = sinx + 5$$

$$y - 5 = sinx$$

$$\sin^{-1}(y-5) = \sin^{-1} sinx$$

$$\sin^{-1}(y-5) = x$$

$$f(x)^{-1} = y = \sin^{-1}(x-5)$$

Domain:

$$-1 \leq (x-5) \leq 1$$

$$-1 + 5 \leq (x) \leq 1 + 5$$

$$4 \leq (x) \leq 6$$

Range

$$\frac{-\pi}{2} \leq \sin^{-1}(x-5) \leq \frac{\pi}{2}$$

Example 8: $If\ f(x) = 4((\sin \pi - 2x) - 1)$

$find\ the\ domain\ and\ range\ for\ f(x)^{-1}$

$$y = 4((\sin \pi - 2x) - 1) \div 4$$

$$\frac{y}{4} = (\sin \pi - 2x) - 1$$

$$\frac{y}{4} + 1 = (\sin \pi - 2x) * \sin^{-1}$$

$$\sin^{-1}(\frac{y}{4} + 1) = \sin^{-1}(\sin \pi - 2x)$$

$$\sin^{-1}(\frac{y}{4} + 1) = (\pi - 2x)$$

$$[\sin^{-1}(\frac{y}{4} + 1) - \pi = -2x] * -1$$

$$2 \div [2x = \pi - \sin^{-1}(\frac{y}{4} + 1)]$$

$$x = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{y}{4} + 1\right)$$

$$f(x)^{-1} = y = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{x}{4} + 1\right)$$

Domain :

$$-1 \leq x\left(\frac{X}{4} + 1\right) \leq 1$$

$$-2 \leq x\,\frac{X}{4} \leq 0$$

$$-8 \leq X \leq 0$$

Range:

$$\frac{-\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$$

$$\frac{-\pi}{2} \leq \sin^{-1}\left(\frac{X}{4} + 1\right) \leq \frac{\pi}{2}$$

$$\frac{-\pi}{4} \leq \frac{1}{2}\sin^{-1}\left(\frac{X}{4} + 1\right) \leq \frac{\pi}{4}$$

$$\frac{\pi}{4} \geq \frac{-1}{2}\sin^{-1}\left(\frac{X}{4} + 1\right) \geq \frac{-\pi}{4}$$

$$\frac{3\pi}{4} \geq \frac{\pi}{2}\,\frac{-1}{2}\sin^{-1}\left(\frac{X}{4} + 1\right) \geq \frac{\pi}{4}$$

H.W find domain of function

1. If $f(x) = \frac{1}{\sin^{-1}(3x+3)}$

2. $f(x) = 2 + \sin^{-1}(3x + 5)$
   $find\ domain\ and\ range\ of\ the\ function$
3. $f(x) = \pi + 7\cos^{-1}(2x - 1)$
   $find\ domain\ and\ range\ of\ the\ function$
4. $\tan^{-1}(\tan \frac{3\pi}{2})$
5. $\sin^{-1}(\sin \frac{13\pi}{6})$

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CHAPTER 25

Derivatives of Inverse Trig Functions

Our goal is simple, and the answers will come quickly. We will derive six new derivative formulas for the six inverse trigonometric functions:

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] \qquad \frac{d}{dx}\left[\tan^{-1}(x)\right] \qquad \frac{d}{dx}\left[\sec^{-1}(x)\right]$$
$$\frac{d}{dx}\left[\cos^{-1}(x)\right] \qquad \frac{d}{dx}\left[\cot^{-1}(x)\right] \qquad \frac{d}{dx}\left[\csc^{-1}(x)\right]$$

These formulas will flow from the inverse rule from Chapter 24 (page 278):

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)}. \tag{25.1}$$

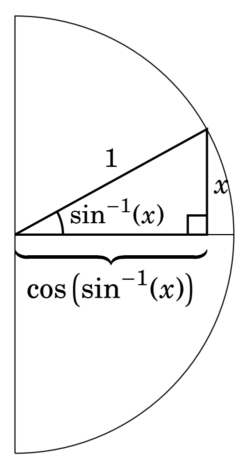
25.1 Derivatives of Inverse Sine and Cosine

Applying the inverse rule (25.1) with $f(x) = \sin(x)$ yields

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\cos\left(\sin^{-1}(x)\right)}. \tag{25.2}$$

We are almost there. We just have to simplify the $\cos\left(\sin^{-1}(x)\right)$ in the denominator. To do this recall

$$\sin^{-1}(x) = \left(\begin{array}{l}\text{the angle } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{for which } \sin(\theta) = x\end{array}\right).$$



Thus $\sin^{-1}(x)$ It is the angle (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) of a the triangle on the unit circle whose opposite side is x. (Because sin of this angle equals x.) Then $\cos\left(\sin^{-1}(x)\right)$ is the length of the adjacent side. By the Pythagorean theorem this side length is $\sqrt{1-x^2}$. Putting $\cos\left(\sin^{-1}(x)\right) = \sqrt{1-x^2}$ into the above Equation (25.2), we get or latest rule:

Rule 20 $\quad \frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$

We reviewed $\sin^{-1}(x)$ In Section 6.1 and presented its graph on page 101. Figure 25.1 repeats the graph, along with the derivative from Rule 20.

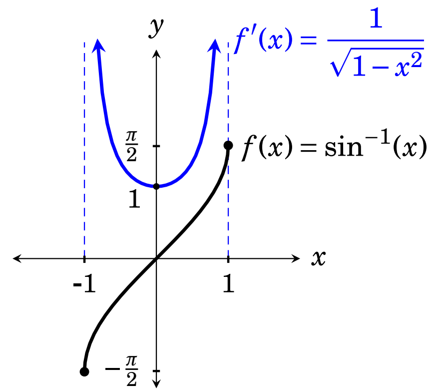


Figure 25.1. The function $\sin^{-1}(x)$ and its derivative. The derivativeis always positive, reflecting the fact that the tangents to $\sin^{-1}(x)$ have positive slope. The derivative has vertical asymptotes at $x = \pm 1$, as the tangents to $\sin^{-1}(x)$ become increasingly steep as x approaches ± 1.

Now consider $\cos^{-1}(x)$. The tangents to its graph (Figure 25.2 below) have *negative* slope, and the geometry suggests that its derivative is *negative* the derivative of $\sin^{-1}(x)$. Indeed this turns out to be exactly the case. This chapter's Exercise 1 asks you to prove our next rule:

Rule 21 $$\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$$

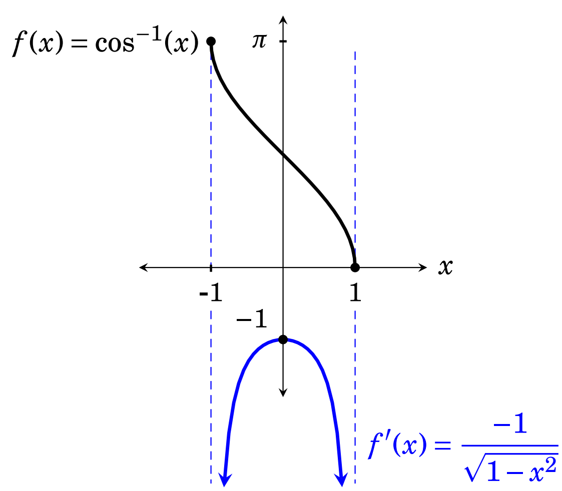


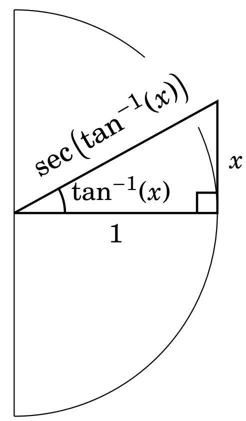
Figure 25.2. The function $\cos^{-1}(x)$ and its derivative.

25.2 Derivatives of Inverse Tangent and Cotangent

Now let's find the derivative of $\tan^{-1}(x)$. Putting $f(x) = \tan(x)$ into the inverse rule (25.1), we have $f^{-1}(x) = \tan^{-1}(x)$ and $f'(x) = \sec^2(x)$, and we get

$$\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{\sec^2\big(\tan^{-1}(x)\big)} = \frac{1}{\Big(\sec\big(\tan^{-1}(x)\big)\Big)^2}. \qquad (25.3)$$

The expression $\sec\big(\tan^{-1}(x)\big)$ in the denominator is the length of the hypotenuse of the triangle to the right. (See example 6.3 in Chapter 6, page 114.) By the Pythagorean theorem, the length is $\sec\big(\tan^{-1}(x)\big) = \sqrt{1+x^2}$. Inserting this into the above Equation (25.4) yields



$$\frac{d}{dx}\Big[\tan^{-1}(x)\Big] = \frac{1}{\Big(\sec\big(\tan^{-1}(x)\big)\Big)^2} = \frac{1}{\big(\sqrt{1+x^2}\big)^2} = \frac{1}{1+x^2}.$$

We now have: **Rule 22** $\dfrac{d}{dx}\Big[\tan^{-1}(x)\Big] = \dfrac{1}{1+x^2}$

We discussed $\tan^{-1}(x)$ in Chapter 6, and its graph is in Figure 6.3. Below Figure 25.3 repeats the graph, along with the derivative $\frac{1}{x^2+1}$.

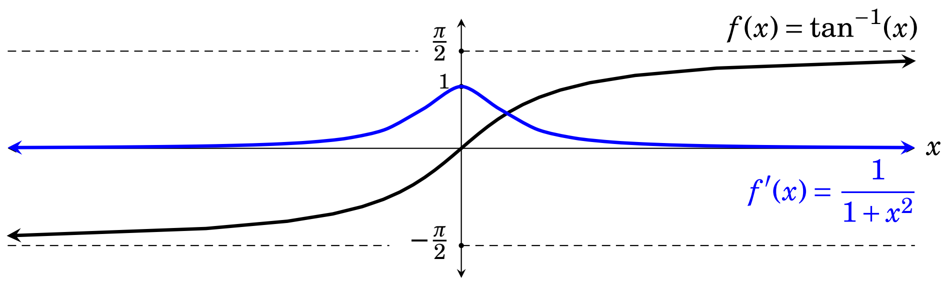


Figure 25.3. The function $\tan^{-1}(x)$ and its derivative $\frac{1}{1+x^2}$. Note $\lim\limits_{x\to\infty} \frac{1}{1+x^2} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{1+x^2} = 0$, reflecting the fact that the tangent lines to $y = \tan^{-1}(x)$ become closer and closer to horizontal as $x \to \pm\infty$. The derivative bumps up to 1 at $x = 0$, where the tangent to $y = \tan^{-1}(x)$ is steepest, with slope 1

Exercise 3 below asks you to mirror the above arguments to deduce:

Rule 23 $\dfrac{d}{dx}\Big[\cot^{-1}(x)\Big] = \dfrac{-1}{1+x^2}$

25.3 Derivatives of Inverse Secant and Cosecant

We reviewed $\sec^{-1}(x)$ in Section 6.3. For its derivative, put $f(x) = \sec(x)$ into the inverse rule (25.1), with $f^{-1}(x) = \sec^{-1}(x)$ and $f'(x) = \sec(x)\tan(x)$. We get

$$\begin{aligned}\frac{d}{dx}\left[\sec^{-1}(x)\right] &= \frac{1}{f'\left(f^{-1}(x)\right)} \\ &= \frac{1}{\sec\left(f^{-1}(x)\right)\cdot\tan\left(f^{-1}(x)\right)} \\ &= \frac{1}{\sec\left(\sec^{-1}(x)\right)\cdot\tan\left(\sec^{-1}(x)\right)}.\end{aligned}$$

Because $\sec\left(\sec^{-1}(x)\right) = x$, the above becomes

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{x\cdot\tan\left(\sec^{-1}(x)\right)}. \qquad (25.4)$$

In Example 6.5 we showed that $\tan\left(\sec^{-1}(x)\right) = \begin{cases} \sqrt{x^2-1} & \text{if } x \text{ is positive} \\ -\sqrt{x^2-1} & \text{if } x \text{ is negative} \end{cases}$
With this, Equation 25.4 above becomes

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x \text{ is positive} \\ \dfrac{1}{-x\sqrt{x^2-1}} & \text{if } x \text{ is negative.} \end{cases}$$

But if x is negative, then $-x$ is positive, and the above consolidates to

Rule 24 $\quad \dfrac{d}{dx}\left[\sec^{-1}(x)\right] = \dfrac{1}{|x|\sqrt{x^2-1}}$

This graph of $\sec^{-1}(x)$ and its derivative is shown in Figure 25.3.

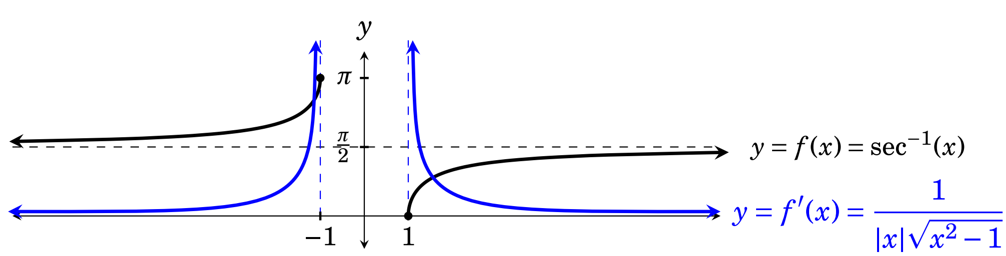


Figure 25.4. The graph of $\sec^{-1}(x)$ and its derivative. The domain of both functions is $(-\infty,-1]\cup[1,\infty)$. Note that the derivative has vertical asymptotes at $x = \pm 1$, where the tangent line to $y = \sec^{-1}(x)$ is vertical.

This chapter's Exercise 2 asks you to use reasoning similar to the above to deduce our final rule.

Rule 25 $\dfrac{d}{dx}\Big[\csc^{-1}(x)\Big] = \dfrac{-1}{|x|\sqrt{x^2-1}}$

Each of our new rules has a chain rule generalization. For example, Rule 25 generalizes as

$$\frac{d}{dx}\Big[\csc^{-1}\big(g(x)\big)\Big] = \frac{-1}{\big|g(x)\big|\sqrt{(g(x))^2-1}}g'(x) = \frac{-g'(x)}{\big|g(x)\big|\sqrt{(g(x))^2-1}}.$$

Here is a summary of this Chapter's new rules, along with their chain rule generalizations.

$\dfrac{d}{dx}\Big[\sin^{-1}(x)\Big] = \dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\Big[\sin^{-1}\big(g(x)\big)\Big] = \dfrac{1}{\sqrt{1-(g(x))^2}}g'(x)$
$\dfrac{d}{dx}\Big[\cos^{-1}(x)\Big] = \dfrac{-1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\Big[\cos^{-1}\big(g(x)\big)\Big] = \dfrac{-1}{\sqrt{1-(g(x))^2}}g'(x)$
$\dfrac{d}{dx}\Big[\tan^{-1}(x)\Big] = \dfrac{1}{1+x^2}$	$\dfrac{d}{dx}\Big[\tan^{-1}\big(g(x)\big)\Big] = \dfrac{1}{1+(g(x))^2}g'(x)$
$\dfrac{d}{dx}\Big[\cot^{-1}(x)\Big] = \dfrac{-1}{1+x^2}$	$\dfrac{d}{dx}\Big[\cot^{-1}\big(g(x)\big)\Big] = \dfrac{-1}{1+(g(x))^2}g'(x)$
$\dfrac{d}{dx}\Big[\sec^{-1}(x)\Big] = \dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\dfrac{d}{dx}\Big[\sec^{-1}\big(g(x)\big)\Big] = \dfrac{g'(x)}{\big\lvert g(x)\big\rvert\sqrt{(g(x))^2-1}}$
$\dfrac{d}{dx}\Big[\csc^{-1}(x)\Big] = \dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\dfrac{d}{dx}\Big[\csc^{-1}\big(g(x)\big)\Big] = \dfrac{-g'(x)}{\big\lvert g(x)\big\rvert\sqrt{(g(x))^2-1}}$

Example 25.1 $\dfrac{d}{dx}\Big[\sqrt{\cos^{-1}(x)}\Big] = \dfrac{d}{dx}\Big[\big(\cos^{-1}(x)\big)^{\frac{1}{2}}\Big] = \dfrac{1}{2}\big(\cos^{-1}(x)\big)^{-\frac{1}{2}}\dfrac{d}{dx}\Big[\cos^{-1}(x)\Big]$

$$= \frac{1}{2}\big(\cos^{-1}(x)\big)^{-\frac{1}{2}}\frac{-1}{\sqrt{1-x^2}} = \boxed{\frac{-1}{2\sqrt{\cos^{-1}(x)}\sqrt{1-x^2}}.}$$

Example 25.2 $\dfrac{d}{dx}\Big[e^{\tan^{-1}(x)}\Big] = e^{\tan^{-1}(x)}\dfrac{d}{dx}\Big[\tan^{-1}(x)\Big] = e^{\tan^{-1}(x)}\dfrac{1}{1+x^2} = \boxed{\dfrac{e^{\tan^{-1}(x)}}{1+x^2}.}$

Example 25.3 $\dfrac{d}{dx}\Big[\tan^{-1}\big(e^x\big)\Big] = \dfrac{1}{1+(e^x)^2}\dfrac{d}{dx}\Big[e^x\Big] = \dfrac{1}{1+e^{2x}}e^x = \boxed{\dfrac{e^x}{1+e^{2x}}.}$

25.4 Summary of Derivative Rules

We have reached the end of our derivative rules. In summary, we have the following rules for specific functions. The corresponding chain rule generalizations are shown to the right.

	Rule	**Chain Rule Generalization**
Constant Rule	$\frac{d}{dx}[c] = 0$	
Identity Rule	$\frac{d}{dx}[x] = 1$	
Power Rule	$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	$\frac{d}{dx}\left[\big(g(x)\big)^n\right] = n\big(g(x)\big)^{n-1}g'(x)$
Exp Rules	$\frac{d}{dx}\left[e^x\right] = e^x$ $\frac{d}{dx}\left[a^x\right] = \ln(a)a^x$	$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}g'(x)$ $\frac{d}{dx}\left[a^{g(x)}\right] = \ln(a)a^{g(x)}g'(x)$
Log Rules	$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$ $\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)}$	$\frac{d}{dx}\left[\ln\big(g(x)\big)\right] = \frac{1}{g(x)}g'(x)$ $\frac{d}{dx}\left[\log_a\big(g(x)\big)\right] = \frac{1}{g(x)\ln(a)}g'(x)$
Trig Rules	$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$ $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$ $\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$ $\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$ $\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$ $\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$	$\frac{d}{dx}\left[\sin\big(g(x)\big)\right] = \cos\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\cos\big(g(x)\big)\right] = -\sin\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\tan\big(g(x)\big)\right] = \sec^2\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\cot\big(g(x)\big)\right] = -\csc^2\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\sec\big(g(x)\big)\right] = \sec\big(g(x)\big)\tan\big(g(x)\big)g'(x)$ $\frac{d}{dx}\left[\csc\big(g(x)\big)\right] = -\csc\big(g(x)\big)\cot\big(g(x)\big)g'(x)$
Inverse Trig Rules	$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$ $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$ $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\sin^{-1}\big(g(x)\big)\right] = \frac{1}{\sqrt{1-(g(x))^2}}g'(x)$ $\frac{d}{dx}\left[\cos^{-1}\big(g(x)\big)\right] = \frac{-1}{\sqrt{1-(g(x))^2}}g'(x)$ $\frac{d}{dx}\left[\tan^{-1}\big(g(x)\big)\right] = \frac{1}{1+(g(x))^2}g'(x)$ $\frac{d}{dx}\left[\cot^{-1}\big(g(x)\big)\right] = \frac{-1}{1+(g(x))^2}g'(x)$ $\frac{d}{dx}\left[\sec^{-1}\big(g(x)\big)\right] = \frac{1}{\lvert g(x)\rvert\sqrt{(g(x))^2-1}}g'(x)$ $\frac{d}{dx}\left[\csc^{-1}\big(g(x)\big)\right] = \frac{-1}{\lvert g(x)\rvert\sqrt{(g(x))^2-1}}g'(x)$

In addition we have the following general rules for the derivatives of combinations of functions.

Constant Multiple Rule: $\frac{d}{dx}\left[\, cf(x)\,\right] = cf'(x)$

Sum/Difference Rule: $\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$

Product Rule: $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Chain Rule: $\frac{d}{dx}\left[\, f\big(g(x)\big)\,\right] = f'(g(x))\,g'(x)$

Inverse Rule: $\frac{d}{dx}\left[\, f^{-1}(x)\,\right] = \frac{1}{f'\left(f^{-1}(x)\right)}$

We used this last rule, the inverse rule, to find the derivatives of $\ln(x)$ and the inverse trig functions. After it has served these purposes it is mostly retired for the remainder of Calculus I, except for the stray exercise or quiz or test question.

This looks like a lot of rules to remember, and it is. But through practice and usage you will reach the point of using them automatically, with hardly a thought. Be sure to get enough practice!

Exercises for Chapter 25

1. Show that $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.

2. Show that $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{|x|\sqrt{x^2-1}}$.

3. Show that $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$.

Find the derivatives of the given functions.

4. $\sin^{-1}\left(\sqrt{2x}\right)$

5. $\ln\left(\tan^{-1}(x)\right)$

6. $e^x \tan^{-1}(x)$

7. $\tan^{-1}(\pi x)$

8. $\sec^{-1}(\pi x)$

9. $\ln\left(\sin^{-1}(x)\right)$

10. $\cos^{-1}(\pi x)$

11. $\sec^{-1}\left(x^5\right)$

12. $e^{\tan^{-1}(\pi x)}$

13. $\tan^{-1}\left(\ln(x)\right) + \pi$

14. $\tan^{-1}\left(x\sin(x)\right)$

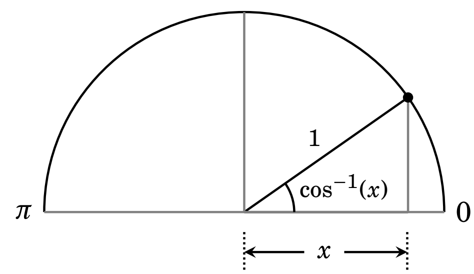
15. $x\sin^{-1}\left(\ln(x)\right)$

Exercise Solutions for Chapter 25

1. Show that $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.

By the inverse rule, $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{1}{-\sin\left(\cos^{-1}(x)\right)}$.

Now we simplify the denominator. From the standard diagram for $\cos^{-1}(x)$ we get $\sin\left(\cos^{-1}(x)\right) = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. With this, the above becomes $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$.



3. Show that $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$.

Suggestion: Verify the identity $\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$. Then differentiate both sides of this.

5. $\frac{d}{dx}\left[\ln\left(\tan^{-1}(x)\right)\right] = \frac{1}{\tan^{-1}(x)}\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{\tan^{-1}(x)}\frac{1}{1+x^2} = \frac{1}{\tan^{-1}(x)\left(1+x^2\right)}$

7. $\frac{d}{dx}\left[\tan^{-1}(\pi x)\right] = \frac{\pi}{1+(\pi x)^2} = \frac{\pi}{1+\pi^2x^2}$

9. $\frac{d}{dx}\left[\ln\left(\sin^{-1}(x)\right)\right] = \frac{1}{\sin^{-1}(x)}\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sin^{-1}(x)}\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}$

11. $\frac{d}{dx}\left[\sec^{-1}\left(x^5\right)\right] = \frac{1}{|x^5|\sqrt{\left(x^5\right)^2-1}}5x^4 = \frac{5x^4}{|x^5|\sqrt{x^{10}-1}} = \frac{5}{|x|\sqrt{x^{10}-1}}$

13. $\frac{d}{dx}\left[\tan^{-1}\left(\ln(x)\right)+\pi\right] = \frac{1}{1+\left(\ln(x)\right)^2}\frac{1}{x} = \frac{1}{x+x\left(\ln(x)\right)^2}$

15. $\frac{d}{dx}\left[x\sin^{-1}\left(\ln(x)\right)\right] = 1\cdot\sin^{-1}\left(\ln(x)\right) + x\frac{d}{dx}\left[\sin^{-1}\left(\ln(x)\right)\right]$

$$= \sin^{-1}\left(\ln(x)\right) + x\frac{1}{\sqrt{1-\left(\ln(x)\right)^2}}\frac{1}{x} = \sin^{-1}\left(\ln(x)\right) + \frac{1}{\sqrt{1-\left(\ln(x)\right)^2}}$$

Find the derivative of functions:

1- $y = \cos^{-1} 5x$

$$y\backslash = \frac{-1}{\sqrt{1-x^2}}$$

$$y\backslash = \frac{-1}{\sqrt{1-(5x)^2}} * 5 = \frac{-5}{\sqrt{1-25x^2}}$$

2- $\mathrm{y} = \arctan(\sqrt[2]{3})$ $\quad \mathrm{y} = \arctan(\mathrm{x}^{\frac{3}{2}})$

$$y\backslash = \frac{1}{1+x^2}$$

$$y\backslash = \frac{1}{1+(x^{\frac{3}{2}})^2} * \frac{2}{3} x^{\frac{1}{2}} \quad y\backslash = \frac{\frac{2}{3}x^{\frac{1}{2}}}{1+x^3}$$

3- $y = 4\cos^{-1} t - 10\tan^{-1} t$

$$y\backslash = \frac{-1}{\sqrt{1-x^2}} - \frac{1}{1+x^2}$$

$$y\backslash = \frac{-4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

4- $y = \sqrt{z}\sin^{-1} z$

$$y\backslash = \sqrt{z} * \frac{1}{\sqrt{1-z^2}} + \sin^{-1} z * \frac{1}{2} z^{\frac{-1}{2}}$$

5- $y = \sin^{-1} 4x^2$

$$y\backslash = \frac{1}{\sqrt{1-(4x)^2}} * 8x = \frac{8x}{\sqrt{1-16x^2}}$$

6- $y = (\sin^{-1} x)^4$

$$y\backslash = 4(\sin^{-1} x)^3 \frac{1}{\sqrt{1-x^2}}$$

7- $y = e^x \tan^{-1} x$

$$y^{\backslash} = e^x \frac{1}{1+x^2} + \tan^{-1} x \tan^{-1} e^x * 1$$

8- $y = \csc^{-1} e^x$

$$y^{\backslash} = \frac{-1}{e^x * \sqrt{(e^x)^2 - 1}} e^x \quad y^{\backslash} = \frac{-1}{\sqrt{(e^x)^2 - 1}}$$

H.W

1- $y = \cos^{-1}\sqrt{2t-1}$

2- $y = \sin^{-1}(2x+1)$

3- $y = \cot^{-1} x - \cot^{-1}\frac{1}{x}$

4- $y = \tan^{-1}\frac{x}{a} + ln\sqrt{\frac{x-a}{x+a}}$

5- $y = \sin^{-1}(sin\frac{\pi}{8})$

6- $y = \sin^{-1}(sin\frac{5\pi}{6})$

7- $y = \sec^{-1}\sqrt{1+x^2}$