Error Coefficients

This method can provide the variation of error with time. It is applicable only to the three standard input signals (step, ramp and parablic) and can be used only for stable systems.

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)\,H(s)}$$, steady state error

$$K_p = \lim_{s \to 0} G(s)\,H(s)$$

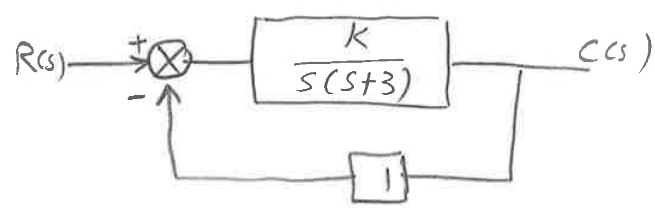
$$K_v = \lim_{s \to 0} s\,G(s)\,H(s)$$

$$K_a = \lim_{s \to 0} s^2\,G(s)\,H(s)$$

Error Coefficients

Ex:1 For input $r(t) = 0.3t$, it is required that steady state error $e_{ss} \leq 0.08$, Determine the value of K for the system

sol



$$G(s) = \frac{K}{s(s+3)} \quad , \quad H(s) = 1$$

$$\therefore G(s)\,H(s) = \frac{K}{s(s+3)}$$

$$r(t) = 0.3t \Rightarrow R(s) = \frac{0.3}{s^2}$$

$$e_{ss} = \lim_{s \to 0} s \frac{R(s)}{1 + G(s)\,H(s)} = \lim_{s \to 0} \cancel{s} \frac{\frac{0.3}{s^{\cancel{2}}}}{1 + \frac{K}{s(s+3)}} = \lim_{s \to 0} \frac{\frac{0.3}{\cancel{s}}}{\frac{s(s+3) + K}{\cancel{s}(s+3)}}$$

$$= \lim_{s \to 0} \frac{0.3\,(s+3)}{s(s+3) + K} \Longrightarrow \because e_{ss} = 0.08$$

$$e_{ss} = 0.08 = \frac{0.9}{K} \Longrightarrow K = \frac{0.9}{0.08} \Longrightarrow K \geq 11.25$$

Ex:2 H-W

A unity feedback system has $G(s) = \frac{K}{s(s+5)(1+0.3s)}$. Determine the steady state error for $r(t) = 7t$, and $K = 10$. Also calculate the value of K for $e_{ss} = 0.25$.

Ex=3 A unity feedback system has $G(s) = \frac{100}{s(s+10)}$ and $r(t) = 3t$

Determin ① the steady state error

② the value of K to reduce the error by 15%.

Sol

$$e_{ss} = \lim_{s \to 0} s \cdot \frac{R(s)}{1 + G(s)H(s)}$$

$$R(s) = \frac{3}{s^2}, \quad G(s)\,H(s) = \frac{100}{s(s+10)}$$

$$\therefore e_{ss} = \lim_{s \to 0} \cancel{s} \cdot \frac{\frac{3}{s^{\cancel{2}}}}{1 + \frac{100}{s(s+10)}} = \lim_{s \to 0} \frac{\frac{3}{\cancel{s}}}{\frac{s(s+10)+100}{\cancel{s}(s+10)}}$$

$$e_{ss} = \lim_{s \to 0} \frac{3(s+10)}{s(s+10)+100} = \frac{30}{100} = 0.3 \quad \underline{\text{ans.}}$$

the new error reduce 15% $\Longrightarrow e_{ss_n} = 0.3 * 0.15 = 0.255$

$$\therefore G(s) = \frac{K}{s(s+10)}$$

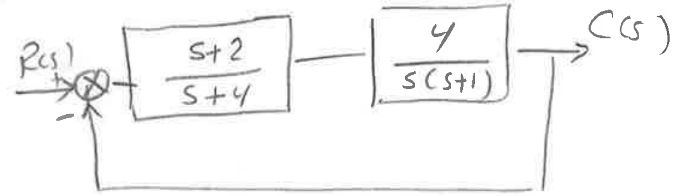
$$e_{ss_n} = \lim_{s \to 0} \cancel{s} \cdot \frac{\frac{3}{s^{\cancel{2}}}}{1 + \frac{K}{s(s+10)}} = \lim_{s \to 0} \frac{3(s+10)}{s(s+10)+K}$$

$$0.225 = \frac{30}{K} \Longrightarrow K = 117.65 \quad \text{ans.}$$

Ex: 4

A stable system shown below in figure

Find ① open loop transfer function
② position, velocity and acceleration error coefficients



Sol

$$\therefore G(s)\,H(s) = \frac{4(s+2)}{s(s+1)(s+4)} \qquad \text{O-L-T-F.}$$

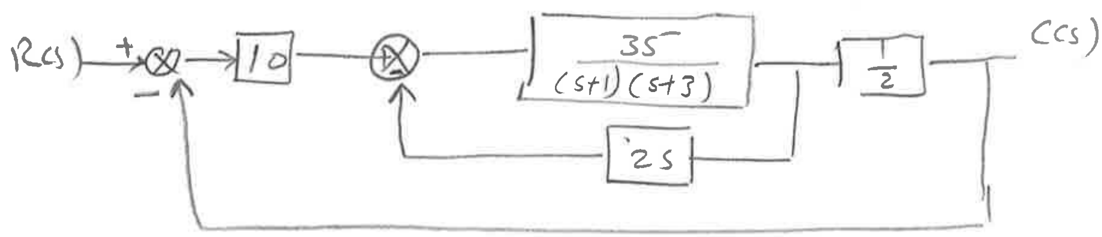
$$K_p = \lim_{s \to 0} G(s)\,H(s) = \frac{8}{0} = \infty$$

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s) = \frac{8}{4} = 2$$

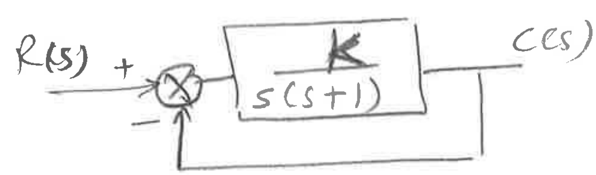
$$K_a = \lim_{s \to 0} s^2 G(s)\,H(s) = 0$$

H-W

Q1/ Determine the error coefficients and steady state error of the system shown for $r(t) = 8t$



Q2/ Determine the range of values K of the system shown, so the steady state error $e_{ss} < 0.004$ when $r(t) = 0.2t$

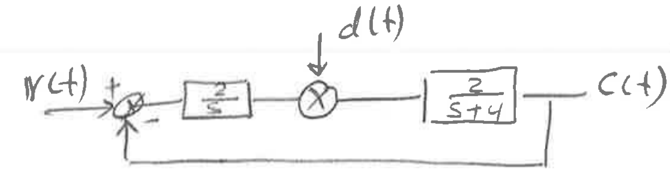


Q3/ Determine the error Coefficients for the system having

$$G(s) = \frac{K}{s^2(s+20)(s+30)}, \quad H(s) = 1$$

Also determine the value of K to limit the steady state error to 0.05 for $r(t) = t^2$

Q4/ For the system shown, determine K_p, K_v and K_a then determine the steady state error to a unit step input, when $d(t) = 0$

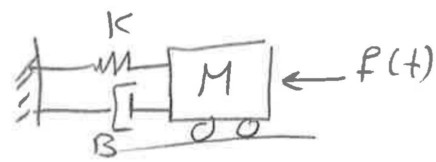


Q5/ A mass-spring-damper system is shown in fig.

a/ write the differential equation

b/ solve this equation when $X(0) = 0$, $\dot{X}(0) = 5$, $f(t) = 0$, $M = 1$

$B = 2$, $K = 2$



Time Domain Analysis

The variation of output with respect to time in known as time response. It is the very important for the design and analysis of Control systems. In time domain systems time is the independent variable.

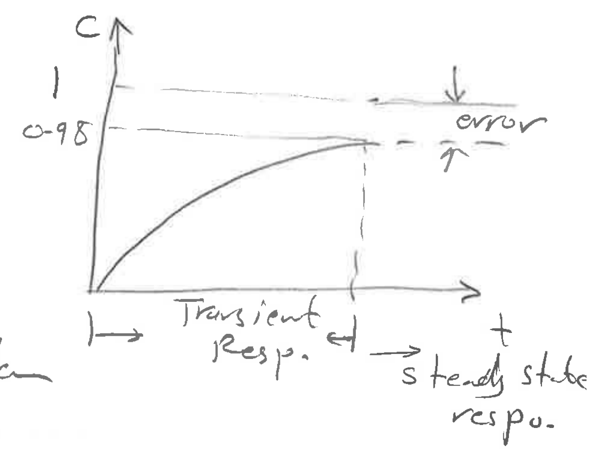
There are two parts of the time response of any time i.e., transient response and steady state response.

Transient response : is the part of the time response which goes to zero after large interval of time.

$$\lim_{t \to \infty} c(t) = 0$$

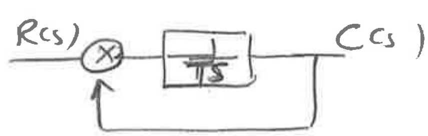
Steady state response : is the part of response that presents even after the transient value die out

$$\therefore C(t) = C_{tr}(t) + C_{ss}$$



- Time Response of First order Control system

$$T.F = \frac{C(s)}{R(s)} = \frac{1}{TS+1}$$



$$\therefore C(s) = \frac{1}{1+TS} R(s)$$

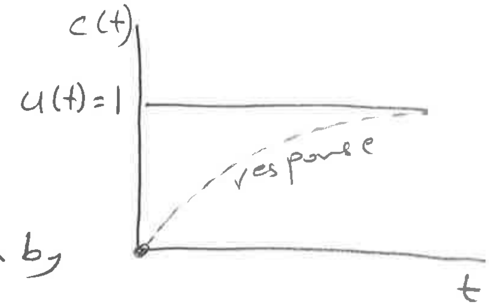
For unit step input , $R(s) = \frac{1}{s}$

$$\therefore C(s) = \frac{1}{s(1+TS)} = \frac{A}{s} + \frac{B}{1+TS}$$, using Partial fraction.

$$1 = A(1+TS) + BS \implies \begin{matrix} 0 = TA + B \\ 1 = A \end{matrix} \implies B = -T$$

$$\therefore C(s) = \frac{1}{s} - \frac{T}{1+TS} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

take inverse Laplace transform of $C(s)$



$$\mathcal{L}^{-1}[C(s)] = 1 - e^{\frac{-t}{T}}$$

the slope of curve $c(t)$ at $t=0$, given by

$$\left.\frac{dC(t)}{dt}\right|_{t=0} = \left.\frac{1}{T} e^{\frac{-t}{T}}\right|_{t=0} = \frac{1}{T}$$

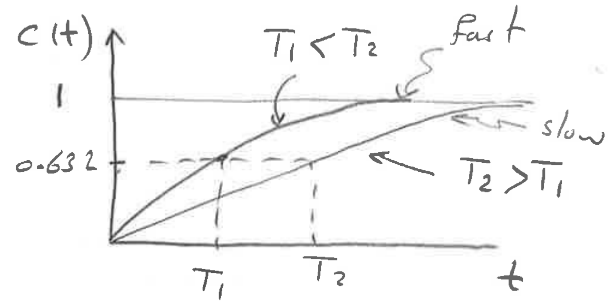
where T is known as Time Constant of the system.

at $t = T \Longrightarrow c(t) = 1 - e^{\frac{-T}{T}} = 1 - e^{-1} = 0.632 = 63.2\%$

Time constant is the time required for the output to rise 63.2% of final or steady state value. It is indicates how fast the system tends to reach the final value.

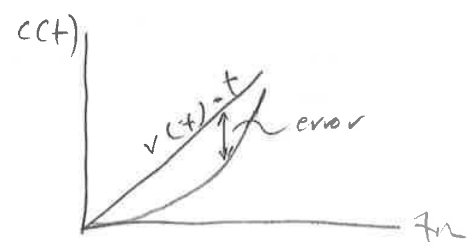
$T \uparrow \longrightarrow$ slow or sluggish system.

$T \downarrow \longrightarrow$ fast response system.



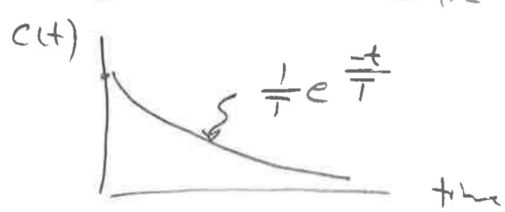
- For ramp input

$$c(t) = t - T(1 - e^{\frac{-t}{T}})$$



- For impulse input

$$C(t) = \frac{1}{T} e^{\frac{-t}{T}}$$



Response of Second Order system to a unit step Input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad , R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{A}{s} + \frac{Bs+C}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\omega_n^2 = A(s^2+2\zeta\omega_n s+\omega_n^2) + (Bs+C)s$$

$$A+B=0$$

$$2\zeta\omega_n A + C = 0$$

$$\omega_n^2 A = \omega_n^2 \Longrightarrow A=1 \quad , B=-1 \quad , C=-2\zeta\omega_n .$$

$$\therefore C(s) = \frac{1}{s} \cancel{+} \frac{-s-2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad , s^2+2\zeta\omega_n s+\omega_n^2 = (s+\zeta\omega_n)^2 + \overbrace{\omega_n^2(1-\zeta^2)}^{\omega_d^2}$$

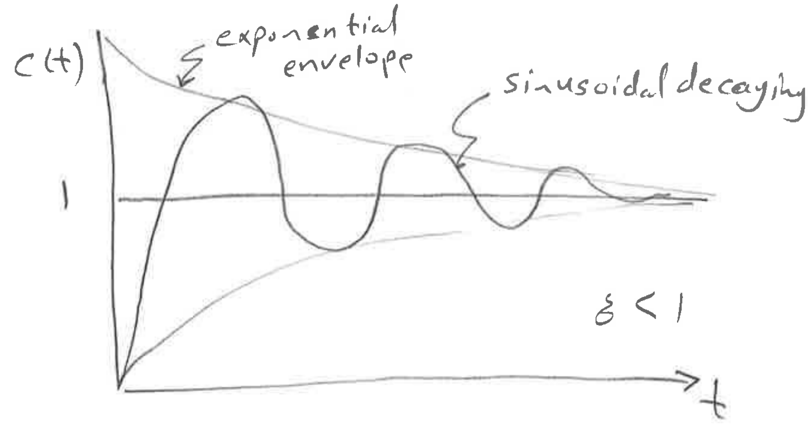
$$= \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{\overbrace{\omega_n\sqrt{1-\zeta^2}}^{\omega_d}}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\omega_d t$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi) \qquad , \phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$



Ex: 1

Determine time response for a unit step input to a unity feedback system having $G(s) = \frac{144}{s(s+12)}$

Sol

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{144}{s(s+12)}}{1+\frac{144}{s(s+12)}} = \frac{\frac{144}{\cancel{s(s+12)}}}{\frac{s(s+12)+144}{\cancel{s(s+12)}}}$$

$$\frac{C(s)}{R(s)} = \frac{144}{s^2+12s+144} \Longrightarrow \omega_n^2 = 144 \Longrightarrow \omega_n = 12 \text{ r/s.}$$

$$2\xi\omega_n = 12 \Longrightarrow 2\xi * 12 = 12$$

$$\therefore \xi = 0.5$$

$$\therefore \omega_d = \omega_n\sqrt{1-\xi^2} = 10.39 \text{ r/s}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = 1.05 \text{ rad}$$

$$\therefore c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)$$

$$c(t) = 1 - \frac{e^{-6t}}{0.866}\sin(10.39\,t + 1.05)$$

Ex: 2

The forward path transfer function of a unity feedback control system is given by $G(s) = \frac{2}{s(s+3)}$, obtain an expression for unit step response of the system.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{2}{s(s+3)+2}$$

$$\therefore C(s) = \frac{2}{s^2+3s+2}R(s) \quad , R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s}\cdot\frac{2}{s^2+3s+2} = \frac{2}{s(s+2)(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$A = 1,\ B = 1,\ C = -2$

$$\therefore C(s) = \frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}$$

$$\therefore C(t) = 1 - 2e^{-t} + e^{-2t}$$

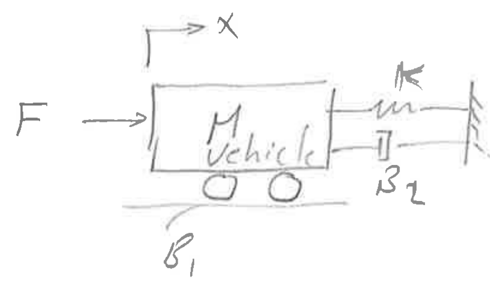
Ex = 3 In figure shown below, $B_1 = 10\ N s/m$, $B_2 = 6\ N\cdot s/m$, $F = 12\ N$, $K = 169\ N/m$, $M = 1\ kg$, calculate

① the final displacement of the vehicle
② the maximum " " "
③ the time required to reach the maximum displacement.

$$F(t) = M\ddot{X} + (B_1 + B_2)\dot{X} + KX$$

$$F(s) = (Ms^2 + (B_1 + B_2)s + K)\, X(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + (B_1 + B_2)s + K}$$



$$\therefore F(s) = \frac{12}{s} \Longrightarrow X(s) = \frac{12}{s(s^2 + 16s + 169)}$$

① the final displacement

$$\lim_{t \to \infty} X(t) = \lim_{s \to 0} s\, X(s) = \lim_{s \to 0} \left[s \cdot \frac{12}{s(s^2 + 16s + 169)} \right]$$

$$= 0.071$$

② $\therefore \omega_n^2 = 169 \Longrightarrow \omega_n = 13\ r/s.$

$$2\zeta\omega_n = 16 \Longrightarrow \zeta = 0.615$$

the max. displacement (overshoot + input)

$$= 1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 1.09\ m.$$

③ time at max. displ. $\Longrightarrow t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.31\ sec.$