Al-Mustaqbal university

Engineering technical college

Department of Building &Construction Engineering

Mathematics

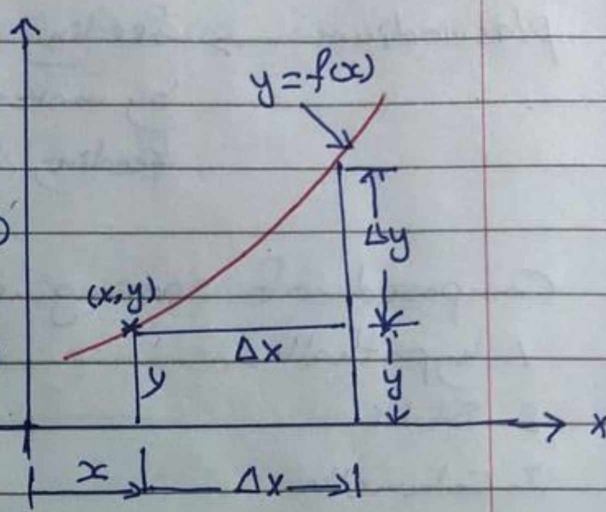
First class

Lecture no.3

Assist. Lecture

Alaa Hussein AbdUlameer

Differentiation المشتقة



$$y = f(x) \quad \text{—} ①$$

at $x + \Delta x$

$$y + \Delta y = f(x + \Delta x) \quad \text{—} ②$$

$2 - 1$

$$\Delta y = f(x + \Delta x) - f(x)$$

divid by Δx to obtain

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example: Derive the function $y = x^2$

$$y = x^2 \quad \text{—} ①$$

$$y + \Delta y = (x + \Delta x)^2$$

subtract

$$y + \Delta y - y = (x + \Delta x)^2 - x^3$$

$$\cancel{y} + \Delta y - \cancel{y} = \cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x^2}$$

$$\Delta y = 2x\Delta x + \Delta x^2$$

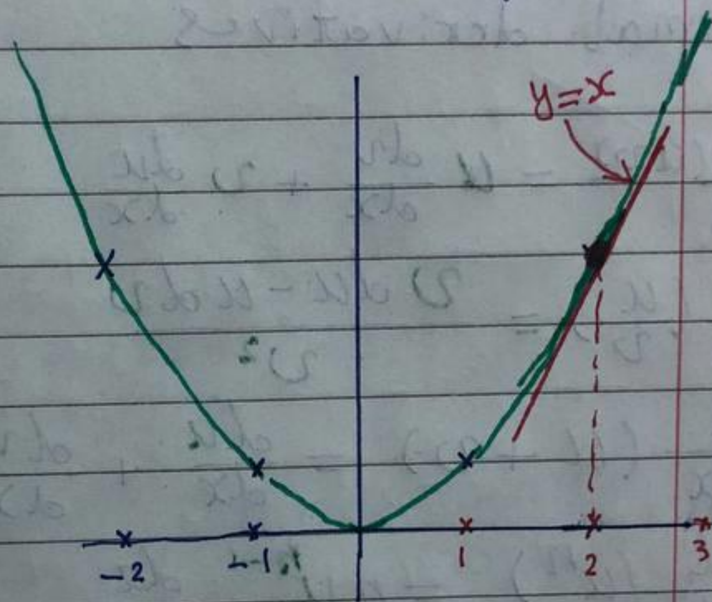
divid both side by Δx

$$\frac{\Delta y}{\Delta x} = \frac{2x\cancel{\Delta x}}{\cancel{\Delta x}} + \frac{\cancel{\Delta x^2}}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \cancel{\Delta x}^{\to 0})$$

$$\frac{dy}{dx} = 2x$$

Ex: Find the equation of a line tangent to the function at $y = x^2$ at a point $x = 2$



Ans:

$$y = x^2$$

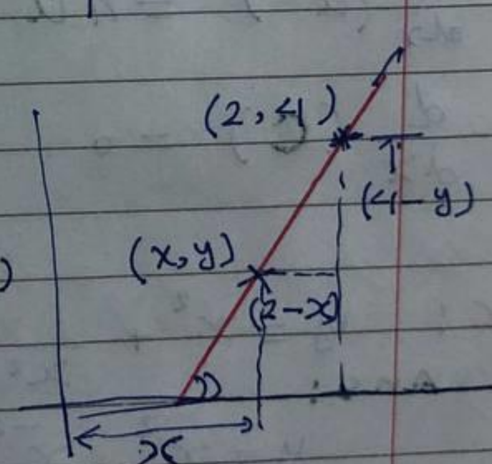
$$\frac{dy}{dx} = 2x$$

$$\therefore m = \frac{dy}{dx} = 2 * 2$$
$$= 4$$

$$y = 2^2 = 4$$

$$m = \frac{4-y}{2-x}$$

$$4 = \frac{4-y}{2-x}$$ (معادلة مستقيم)



$$4 - y = 4(2-x)$$
$$4 - y = 8 - 4x$$
$$-y = 8 - 4 - 4x$$
$$-y = 4 - 4x$$
$$y = 4x - 4$$

Ex: Find $\frac{dy}{dx}$ if $y = x^3 + 7x^2 - 5x + 4$

Ans: $\frac{dy}{dx} = 3x^2 + 14x - 5$

Formal derivatives

1) $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

2) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

3) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

4) $\frac{d}{dx}(u^n) = n\,u^{n-1}\frac{du}{dx}$

5) $\frac{d}{dx}(C) = 0$ where C: Constant

Ex: $y = x^2 + \frac{1}{x^2}$, $x \neq 0$

Ans:

$$y = x^2 + x^{-2}$$

$$\therefore \frac{dy}{dx} = 2x - 2x^{-3}$$

Problems: Find $\frac{dy}{dx}$

1) $y = (x^2+1)^5$

Ans:

$$\frac{dy}{dx} = 5(x^2+1)^4 * 2x = 10x(x^2+1)^4$$

2) $y = \frac{2x+5}{3x-2}$

Ans:

$$\frac{dy}{dx} = \frac{(3x-2)*2 - (2x+5)*3}{(3x-2)^2}$$

$$= \frac{\cancel{6x} - 4 - \cancel{6x} - 15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$$

3) $y = (x-1)(x+2)$

Ans:

$$\frac{dy}{dx} = (x-1)*1 + (x+2)*1$$

$$= x-1 + x+2$$

$$= 2x+1$$

Find $\frac{ds}{dt}$ in each of the following problems

$$s = \frac{t}{t^2+1}$$

Ans: $\frac{ds}{dt} = \frac{(t^2+1) - t*2t}{(t^2+1)^2} = \frac{t^2+1-2t^2}{(t^2+1)} = \frac{(1-t^2)}{(1+t^2)}$

Ex: Find $\frac{dy}{dx}$ if $x^5 + 4xy^3 - 3y^5 = 2$

Ans:

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(4xy^3) - \frac{d}{dx}(3y^5) = \frac{d}{dx}(2)$$

$$5x^4 \cancel{\frac{dx}{dx}} + 4\left(x * 3y^2 \frac{dy}{dx} + y^3 \cancel{\frac{dx}{dx}}\right) - 3*5y^4 \frac{dy}{dx} = 0$$

$$5x^4 + 12xy^2 \frac{dy}{dx} + 4y^3 - 15y^4 \frac{dy}{dx} = 0$$

$$5x^4 + 4y^3 = (-12xy^2 + 15y^4)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{5x^4 + 4y^3}{-12xy^2 + 15y^4}$$

Ex Find $\frac{dy}{dx}$ for the implicit relation

$$x^2 y + x y^2 = 6$$

Ans:

$$2x \cancel{\frac{dx}{dx}}^{1} y + x^2 \frac{dy}{dx} + \cancel{\frac{dx}{dx}}^{1} * y^2 + x * 2y \frac{dy}{dx} = 0$$

$$2xy + x^2 \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} = 0$$

$$2xy + y^2 = -x^3 \frac{dy}{dx} - 2xy \frac{dy}{dx}$$

$$(2xy + y^2) = -(x^3 + 2xy)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{-(2xy + y^2)}{(x^2 + 2xy)}$$

Ex: $2xy + y^2 = x + y$

$$2\left[x\frac{dy}{dx} + y\cancel{\frac{dx}{dx}}^{1}\right] + 2y\frac{dy}{dx} = \cancel{\frac{dx}{dx}}^{1} + \frac{dy}{dx}$$

$$2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = \frac{dy}{dx} + 1$$

$$2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$$

$$\therefore \frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$$

$$\frac{dy}{dx} = \frac{1 - 2y}{2x + 2y - 1}$$

Exercises 2.6

Derivatives of Rational Powers

Find dy/dx in Exercises 1–10.

1. $y = x^{9/4}$
2. $y = x^{-3/5}$
3. $y = \sqrt[3]{2x}$
4. $y = \sqrt[4]{5x}$
5. $y = 7\sqrt{x+6}$
6. $y = -2\sqrt{x-1}$
7. $y = (2x+5)^{-1/2}$
8. $y = (1-6x)^{2/3}$
9. $y = x(x^2+1)^{1/2}$
10. $y = x(x^2+1)^{-1/2}$

Find the first derivatives of the functions in Exercises 11–18.

11. $s = \sqrt[7]{t^2}$
12. $r = \sqrt[4]{\theta^{-3}}$
13. $y = \sin\left[(2t+5)^{-2/3}\right]$
14. $z = \cos\left[(1-6t)^{2/3}\right]$
15. $f(x) = \sqrt{1-\sqrt{x}}$
16. $g(x) = 2(2x^{-1/2}+1)^{-1/3}$
17. $h(\theta) = \sqrt[3]{1+\cos(2\theta)}$
18. $k(\theta) = (\sin(\theta+5))^{5/4}$

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 19–32.

19. $x^2y + xy^2 = 6$
20. $x^3 + y^3 = 18xy$
21. $2xy + y^2 = x + y$
22. $x^3 - xy + y^3 = 1$
23. $x^2(x-y)^2 = x^2 - y^2$
24. $(3xy+7)^2 = 6y$
25. $y^2 = \dfrac{x-1}{x+1}$
26. $x^2 = \dfrac{x-y}{x+y}$
27. $x = \tan y$
28. $x = \sin y$
29. $x + \tan(xy) = 0$
30. $x + \sin y = xy$
31. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
32. $y^2 \cos\left(\dfrac{1}{y}\right) = 2x + 2y$

Find $dr/d\theta$ in Exercises 33–36.

33. $\theta^{1/2} + r^{1/2} = 1$
34. $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$
35. $\sin(r\theta) = \dfrac{1}{2}$
36. $\cos r + \cos\theta = r\theta$

Higher Derivatives

In Exercises 37–42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

37. $x^2 + y^2 = 1$
38. $x^{2/3} + y^{2/3} = 1$
39. $y^2 = x^2 + 2x$
40. $y^2 - 2x = 1 - 2y$
41. $2\sqrt{y} = x - y$
42. $xy + y^2 = 1$
43. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
44. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

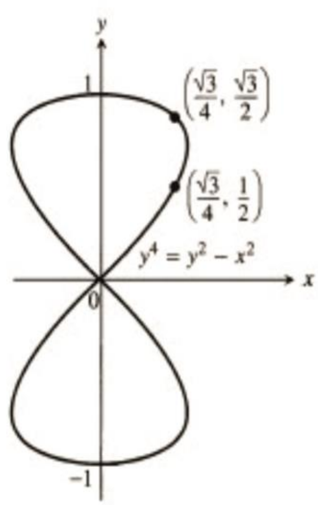
Slopes, Tangents, and Normals

In Exercises 45 and 46, find the slope of the curve at the given points.

45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
46. $(x^2+y^2)^2 = (x-y)^2$ at $(1, 0)$ and $(1, -1)$

In Exercises 47–56, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

47. $x^2 + xy - y^2 = 1$, $(2, 3)$
48. $x^2 + y^2 = 25$, $(3, -4)$
49. $x^2y^2 = 9$, $(-1, 3)$
50. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$
51. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$
52. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$
53. $2xy + \pi \sin y = 2\pi$, $(1, \pi/2)$
54. $x \sin 2y = y \cos 2x$, $(\pi/4, \pi/2)$
55. $y = 2\sin(\pi x - y)$, $(1, 0)$
56. $x^2\cos^2 y - \sin y = 0$, $(0, \pi)$
57. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?
58. Find points on the curve $x^2 + xy + y^2 = 7$ (a) where the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis. In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?
59. *The eight curve.* Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



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Lecture No.5

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Chain Rule :

The chain rule is used to differentiate the composite functions.

1. Chain rule for function of single variable defined along paths.

$$y = f(x)$$

$$x = x(t) \quad \Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx} * \frac{dx}{dt}$$

$$y = f[x(t)]$$

Ex: if $y = x^2+1$, $x = \tan^{-1} t$, find $\frac{dy}{dt}$?

Sol:
$$\frac{dy}{dx} = 2x$$

$$\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= 2x \cdot \frac{1}{1+t^2} = 2\tan^{-1} t * \frac{1}{1+t^2}$$

2. For the function $Z = f(x,y)$ of two variable defined along path.

$$Z = f[x(t), g(t)]$$

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} + \frac{dy}{dt}$$

Ex: $Z = f(x,y) = x^2 y^3$, $x = \cos t$, $y = \sin t$ find $\frac{df}{dt}$?

Sol:

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

$$f_x = y^3 2x = 2xy^3$$

$$f_y = 3y^2 x^2$$

$$\frac{dy}{dt} = \cos t \quad , \quad \frac{dx}{dt} = -\sin t$$

$$\frac{df}{dt} = 2xy^3(-\sin t) + 3y^2x^2(\cos t)$$

$$= 2\cos t(\sin t)^3(-\sin t) + 3(\sin t)^2(\cos t)^2(\cos t)$$

$$= -2\cos t(\sin t)^4 + 3(\sin t)^2(\cos t)^3$$

3. chain rule for function to more than two variable defined along path:

$$w = f(x_1, x_2, x_3, \dots, x_n)$$

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \dots + \frac{df}{dx_n} \cdot \frac{dx_n}{dt}$$

Ex: let $w = xy \sin z$ where $x = \cos t$, $y = \sin t$, $z = 1+t^2$

find $\frac{dw}{dt}$?

Sol:

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \frac{df}{dx_3} \cdot \frac{dx_3}{dt}$$

$F_x = y \sin z$, $F_y = x \sin z$, $F_z = xy \cos z$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{dz}{dt} = 2t$$

$$\frac{dw}{dt} = y \sin z(-\sin t) + x \sin z(\cos t) + xy \cos z\,(2t)$$

$$= y \sin(1+t^2)(-\sin t) + \cos t \sin(1+t^2)\,x + \cos t \sin t \cos(1+t^2)$$

$$(2t)$$

4. Chain rule for function of two variable defined on surface :

If $Z = f(x,y)$ has partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and if

$X = x(r,s)$ and $y = y(r,s)$

$Z = f[x(r,s), y(r,s)]$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Ex : find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ if $z = f(x,y) = x^2 + y^2$

$X = r + e^s$, $y = \ln s$?

Sol :

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$f_x = 2x$, $\frac{\partial x}{\partial r} = 1$, $\frac{\partial y}{\partial r} = 0$

$f_y = 2y$, $\frac{\partial x}{\partial s} = e^s$, $\frac{\partial y}{\partial s} = \frac{1}{s}$

$$\frac{\partial z}{\partial r} = 2x(1) + 2y(0) = 2x = 2(r + e^s)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2x e^s + 2y (1/s)$$

$$= 2(r+e^s) \cdot e^s + 2(\ln s) \cdot \frac{1}{s}$$

$$= 2re^s + 2e^{2s} + \frac{2}{s} \ln s$$

Ex: If Z is a differentiable function of x and y which Satisfy the equation $x^3 + y^3 + z^3 + 3x^2 \sin y \tan z = 5$? find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$?

Sol:

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 3 \sin y \left(x^2 \sec^2 z \frac{\partial z}{\partial x} + \tan z * 2x \right) = 0$$

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 3x^2 \sin y \sec^2 z \frac{\partial z}{\partial x} + 6x \sin y \tan z = 0$$

$$3x^2 \sin y \sec^2 z \frac{\partial z}{\partial x} + 3z^2 \frac{\partial z}{\partial x} = -3x^2 - 6x \sin y \tan z$$

$$\frac{\partial z}{\partial x} (3x^2 \sin y \sec^2 z + 3z^2) = -3x^2 - 6x \sin y \tan z$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6x \sin y \tan z}{3x^2 \sin y \sec^2 z + 3z^2}$$

$$0+3y^2+3z^2\frac{\partial z}{\partial y}+3x^2\left(\sin y\ \sec^2 z\ \frac{\partial z}{\partial y}+\tan z\ \cos y\right)=0$$

$$3y^2+3z^2\frac{\partial z}{\partial y}+3x^2\sin y\ \sec^2 z\ \frac{\partial z}{\partial y}+3x^2\tan z\ \cos y=0$$

$$3z^2\frac{\partial z}{\partial y}+3x^2\sin y\ \sec^2 z\ \frac{\partial z}{\partial y}=-3y^2-3x^2\tan z\ \cos y$$

$$\frac{\partial z}{\partial y}\left(3z^2+3x^2\sin y\ \sec^2 z\right)=-3y^2-3x^2\tan z\ \cos y$$

$$\frac{\partial z}{\partial y}=\frac{-3y^2-3x^2\tan z\ \cos y}{3z^2+3x^2\sin y\ \sec^2 z}$$

5. chain rule for function of three variables defined on surface :-

if $w=f(x,y,z)$ has partial derivatives

$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}$ and if $x=x(r,s)$, $z=z(r,s)$

also have partial derivatives

$$w=f\left[x(r,s),y(r,s),z(r,s)\right]$$

$$\frac{\partial w}{\partial r}=\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Ex: find $\frac{\partial w}{\partial r}$ if $w = f(x,y,z) = x + 2y + z^2$ where $x = \frac{r}{s}$

$y = r^2 + e^s$, $z = 2r$?

Sol:

$$w = x + 2y + z^2$$

$$\frac{\partial w}{\partial x} = 1 + 0 + 0 = 1$$

$$\frac{\partial w}{\partial y} = 0 + 2 + 0 = 2$$

$$\frac{\partial w}{\partial z} = 0 + 0 + 2z = 2z$$

$$x = \frac{r}{s} \Rightarrow \frac{\partial x}{\partial r} = \frac{1}{s} * 1 = \frac{1}{s}$$

$$y = r^2 + e^s \Rightarrow \frac{\partial y}{\partial r} = 2r + 0 = 2r$$

$$z = 2r \Rightarrow \frac{\partial z}{\partial r} = 2$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= 1 * \frac{1}{s} + 2 * 2r + 2z * 2$$

$$= \frac{1}{s} + 4r + 4z$$

$$= \frac{1}{s} + 4r + 4(2r) = \frac{1}{s} + 12r$$

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Lecture six

Higher order partial derivative :

- 2^{nd} Order partial derivative for functions with two variable $F(x,y)$ has partial derivative $F_x = \frac{\partial f}{\partial x}$ and $F_y = \frac{\partial f}{\partial y}$.

- The 2^{nd} order partial derivative are denoted by:

$$F_{xx} = \frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial F_x}{\partial x}$$

$$F_{yy} = \frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial F_y}{\partial y}$$

$$F_{xy} = \frac{\partial^2 F}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f_x}{\partial y}$$

$$F_{yx} = \frac{\partial^2 F}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial F_y}{\partial x}$$

Ex : let $F(x,y) = x\cos y + y e^x$ find F_{xx}, F_{yy}, F_{xy}, F_{yx} ?

Sol:

1- $F_{xx} = \cos y * 1 + y e^x * 1$

$= \cos y + y e^x$

$= 0 + y e^x * 1 = y e^x$

2- $F_{yy} = X(-\sin y) + e^X * 1$

$= -X \sin y + e^X$

$= -X \cos y + 0$

$= -X \cos y$

3- $F_{xy} = \cos y + y e^X$

$= -\sin y + e^X * 1$

$= -\sin y + e^X$

4- $F_{yx} = -X \sin y + e^X$

$= -\sin y (1) + e^X (1)$

$= -\sin y + e^X$

- Consider the 3rd order partial derivative of the function $Z = f(x,y)$.

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f_x}{\partial x^2}\right) = \frac{\partial f_{xx}}{\partial x} = f_{xxx} \quad \text{Similarity}$$

$$\frac{\partial^3 f}{\partial y^3} = f_{yyy}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial f_{xx}}{\partial y} = f_{xxy}$$

Ex: $f(x,y) = x\cos y + y e^x$ find f_{xxx}, f_{yyy}, f_{xxy}, f_{yyx} ?

Sol:

$$f_{xxx} = \cos y * 1 + y e^x * 1$$

$$= \cos y + y e^x$$

$$= 0 + y * e^x * 1 = y e^x$$

$$= y e^x * 1 = y e^x$$

$$f_{yyy} = x(-\sin y) + e^x * 1$$

$$= -x \sin y + e^x = -x \cos y + 0$$

$$= -x \cos y = -x(-\sin y) = x \sin y$$

H.W $\Rightarrow$ f_{xxy}, f_{yyx} ?

second & high order derivatives

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$y' = \frac{dy}{dx}$ is the first Order derivatives of y to x

if y' is defferentiable with respect to x, so

$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ which is call

second order derivative ($\frac{d^2y}{dx^2}$ or y'')

again if y'' is a differentiable in x, so

$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ is the third order derivative

In general

$$y^{(n)} = \frac{d}{dx} y^{(n-1)}$$

How to read derivative Symbols

y' "y prime"

y'' "y double prime"

y''' "y triple prime"

$y^{(n)}$ "y super n"

Ex: find y', y'', y''' and $y^{(4)}$ for $y = x^3 - 3x^2 + 2$

Ans:

$y' = 3x^2 - 6x$, $y'' = 6x - 6$

$y''' = 6$ $y^{(4)} = 0$

High order derivatives

let $y = f(x)$

1^{st} order derivative $\quad y' = f'(x)$

2^{nd} " " $\quad y'' = f''(x)$

3^{rd} " " $\quad y''' = f'''(x)$

Ex: $y = x^4 + 3x^3 + 2x + 5$

Find y', y'' and y'''

Ans:

$$\frac{dy}{dx} = y' = 4x^3 + 9x^2 + 2$$

$$\frac{d^2y}{dx^2} = y'' = 12x^2 + 18x$$

$$\frac{d^3y}{dx^3} = y''' = 24x + 18$$

Ex: if $y = x(x+3)^2$, Find y', y'', y'''

Ans

$$y' = x * 2(x+3) + (x+3)$$
$$= 2x^2 + 3x + x + 3$$
$$= 2x^2 + 4x + 3$$
$$y'' = 4x^2 + 4$$
$$y''' = 12x$$

High order derivatives of rational function

Ex: Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$

Ans:

$$2x^3 - 3y^2 = 7$$

$$6x^2 - 6y\frac{dy}{dx} = 0 \quad \text{or can be written}$$

$$6x^2 - yy' = 0 \longrightarrow \left(y' = \frac{6x^2}{y}\right)$$

derive again

$$12x - (y * y'' + y' * y') = 0$$

$$12x - yy'' + y'^2 = 0$$

$$-yy'' = -y'^2 - 12x$$

$$yy'' = y'^2 + 12x$$

$$y'' = \frac{(y')^2 + 12x}{y} = \frac{\left(\frac{6x^2}{y}\right)^2 + 12x}{y}$$

$$= \frac{\frac{36x^4}{y^2} + 12x}{y} = \frac{36x^4\ 12xy^2}{y^3}$$

Exercises 2.6

Derivatives of Rational Powers

Find dy/dx in Exercises 1–10.

1. $y = x^{9/4}$ **2.** $y = x^{-3/5}$

3. $y = \sqrt[3]{2x}$ **4.** $y = \sqrt[4]{5x}$

5. $y = 7\sqrt{x+6}$ **6.** $y = -2\sqrt{x-1}$

7. $y = (2x+5)^{-1/2}$ **8.** $y = (1-6x)^{2/3}$

9. $y = x(x^2+1)^{1/2}$ **10.** $y = x(x^2+1)^{-1/2}$

Find the first derivatives of the functions in Exercises 11–18.

11. $s = \sqrt[7]{t^2}$ **12.** $r = \sqrt[4]{\theta^{-3}}$

13. $y = \sin\left[(2t+5)^{-2/3}\right]$ **14.** $z = \cos\left[(1-6t)^{2/3}\right]$

15. $f(x) = \sqrt{1-\sqrt{x}}$ **16.** $g(x) = 2(2x^{-1/2}+1)^{-1/3}$

17. $h(\theta) = \sqrt[3]{1+\cos(2\theta)}$ **18.** $k(\theta) = (\sin(\theta+5))^{5/4}$

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 19–32.

19. $x^2y + xy^2 = 6$ **20.** $x^3 + y^3 = 18xy$

21. $2xy + y^2 = x + y$ **22.** $x^3 - xy + y^3 = 1$

23. $x^2(x-y)^2 = x^2 - y^2$ **24.** $(3xy+7)^2 = 6y$

25. $y^2 = \dfrac{x-1}{x+1}$ **26.** $x^2 = \dfrac{x-y}{x+y}$

27. $x = \tan y$ **28.** $x = \sin y$

29. $x + \tan(xy) = 0$ **30.** $x + \sin y = xy$

31. $y\sin\left(\dfrac{1}{y}\right) = 1 - xy$ **32.** $y^2\cos\left(\dfrac{1}{y}\right) = 2x + 2y$

Find $dr/d\theta$ in Exercises 33–36.

33. $\theta^{1/2} + r^{1/2} = 1$ **34.** $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$

35. $\sin(r\theta) = \dfrac{1}{2}$ **36.** $\cos r + \cos\theta = r\theta$

Higher Derivatives

In Exercises 37–42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

37. $x^2 + y^2 = 1$ **38.** $x^{2/3} + y^{2/3} = 1$

39. $y^2 = x^2 + 2x$ **40.** $y^2 - 2x = 1 - 2y$

41. $2\sqrt{y} = x - y$ **42.** $xy + y^2 = 1$

43. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.

44. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

Solve all

Slopes, Tangents, and Normals

In Exercises 45 and 46, find the slope of the curve at the given points.

45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

46. $(x^2+y^2)^2 = (x-y)^2$ at $(1, 0)$ and $(1, -1)$

In Exercises 47–56, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

47. $x^2 + xy - y^2 = 1$, $(2, 3)$

48. $x^2 + y^2 = 25$, $(3, -4)$

49. $x^2y^2 = 9$, $(-1, 3)$

50. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$

51. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$

52. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$

53. $2xy + \pi\sin y = 2\pi$, $(1, \pi/2)$

54. $x\sin 2y = y\cos 2x$, $(\pi/4, \pi/2)$

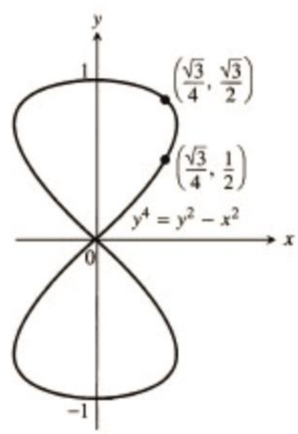
55. $y = 2\sin(\pi x - y)$, $(1, 0)$

56. $x^2\cos^2 y - \sin y = 0$, $(0, \pi)$

57. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

58. Find points on the curve $x^2 + xy + y^2 = 7$ (a) where the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis. In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?

59. *The eight curve.* Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



How to read the symbols for derivatives

y' "y prime" y'' "y double prime"

$\dfrac{d^2y}{dx^2}$ "d squared y dx squared"

y''' "y triple prime"

$y^{(n)}$ "y super n"

$\dfrac{d^n y}{dx^n}$ "d to the n of y by dx to the n"

EXAMPLE 13 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

The function has derivatives of all orders, the fifth and later derivatives all being zero. ❑

Solve as much as you can

Exercises 2.2

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \dfrac{4x^3}{3} - x$
6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$
7. $w = 3z^{-2} - \dfrac{1}{z}$
8. $s = -2t^{-1} + \dfrac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$
12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

In Exercises 13–16, find y' (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (x - 1)(x^2 + x + 1)$
15. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$
16. $y = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x} + 1\right)$

Find the derivatives of the functions in Exercises 17–28.

17. $y = \dfrac{2x + 5}{3x - 2}$
18. $z = \dfrac{2x + 1}{x^2 - 1}$
19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$
20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$
21. $v = (1 - t)(1 + t^2)^{-1}$
22. $w = (2x - 7)^{-1}(x + 5)$
23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$
24. $u = \dfrac{5x + 1}{2\sqrt{x}}$
25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$
26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$
28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the derivatives of all orders of the functions in Exercises 29 and 30.

29. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$
30. $y = \dfrac{x^5}{120}$

Find the first and second derivatives of the functions in Exercises 31–38.

31. $y = \dfrac{x^3 + 7}{x}$
32. $s = \dfrac{t^2 + 5t - 1}{t^2}$
33. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
34. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$
35. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$
36. $w = (z + 1)(z - 1)(z^2 + 1)$
37. $p = \left(\dfrac{q^2 + 3}{12q}\right)\left(\dfrac{q^4 - 1}{q^3}\right)$
38. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

Using Numerical Values

39. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c)** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(7v - 2u)$

40. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c)** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(7v - 2u)$

Section 2.2, pp. 129–131

1. $\frac{dy}{dx} = -2x,\ \frac{d^2y}{dx^2} = -2$

3. $\frac{ds}{dt} = 15t^2 - 15t^4,\ \frac{d^2s}{dt^2} = 30t - 60t^3$

5. $\frac{dy}{dx} = 4x^2 - 1,\ \frac{d^2y}{dx^2} = 8x$

7. $\frac{dw}{dz} = -6z^{-3} + \frac{1}{z^2},\ \frac{d^2w}{dz^2} = 18z^{-4} - \frac{2}{z^3}$

9. $\frac{dy}{dx} = 12x - 10 + 10x^{-3},\ \frac{d^2y}{dx^2} = 12 - 30x^{-4}$

11. $\frac{dr}{ds} = \frac{-2}{3s^3} + \frac{5}{2s^2},\ \frac{d^2r}{ds^2} = \frac{2}{s^4} - \frac{5}{s^3}$

13. $y' = -5x^4 + 12x^2 - 2x - 3$ **15.** $y' = 3x^2 + 10x + 2 - \frac{1}{x^2}$

17. $y' = \frac{-19}{(3x-2)^2}$ **19.** $g'(x) = \frac{x^2 + x + 4}{(x + 0.5)^2}$

21. $\frac{dv}{dt} = \frac{t^2 - 2t - 1}{(1 + t^2)^2}$ **23.** $f'(s) = \frac{1}{\sqrt{s}(\sqrt{s} + 1)^2}$

25. $v' = -\frac{1}{x^2} + 2x^{-3/2}$ **27.** $y' = \frac{-4x^3 - 3x^2 + 1}{(x^2 - 1)^2(x^2 + x + 1)^2}$

29. $y' = 2x^3 - 3x - 1,\ y'' = 6x^2 - 3,\ y''' = 12x,\ y^{(4)} = 12,$ $y^{(n)} = 0$ for $n \geq 5$

31. $y' = 2x - 7x^{-2},\ y'' = 2 + 14x^{-3}$

33. $\frac{dr}{d\theta} = 3\theta^{-4},\ \frac{d^2r}{d\theta^2} = -12\theta^{-5}$

35. $\frac{dw}{dz} = -z^{-2} - 1,\ \frac{d^2w}{dz^2} = 2z^{-3}$

37. $\frac{dp}{dq} = \frac{1}{6}q + \frac{1}{6}q^{-3} + q^{-5},\ \frac{d^2p}{dq^2} = \frac{1}{6} - \frac{1}{2}q^{-4} - 5q^{-6}$

39. a) 13 b) −7 c) 7/25 d) 20 **41.** a) $y = -\frac{x}{8} + \frac{5}{4}$ b) $m = -4$ at (0, 1) c) $y = 8x - 15, y = 8x + 17$

43. $y = 4x, y = 2$ **45.** $a = 1, b = 1, c = 0$ **47.** a) $y = 2x + 2$, c) (2, 6) **49.** $\frac{dP}{dV} = -\frac{nRT}{(V - nb)^2} + \frac{2an^2}{V^3}$

51. The Product Rule is then the Constant Multiple Rule, so the latter is a special case of the Product Rule.

55. a) $\frac{3}{2}x^{1/2}$, b) $\frac{5}{2}x^{3/2}$, c) $\frac{7}{2}x^{5/2}$, d) $\frac{d}{dx}\left(x^{n/2}\right) = \frac{n}{2}x^{(n/2)-1}$