Al-Mustaqbal university

Engineering technical college

Department of Building &Construction Engineering

Mathematics

First class

Lecture No.11

Define integral

Assist. Lecture

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Define Integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

* التكامل يتوزع على الجمع والطرح فقط

$$\int k\, f(x)\,dx = k \int f(x)\,dx$$

constant

$$\int \sin^m(x)\,dx$$

m زوجي

use :

1- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

2- $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

m فردي

1- Sperate one(Sin)

2- $\sin^2 x = 1 - \cos^2 x$

* $\sin^2 x + \cos^2 x = 1$

3- let $u = \cos x$

Ex: $\int \sin^2(x)\,dx$

$$= \int \frac{1}{2}(1 - \cos(2x))\,dx$$

$$= \frac{1}{2} \int (1 - \cos(2x))\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

Ex: $\int \sin^3 x \, dx$

$= \int \sin x \cdot \sin^2 x \, dx$

$= \int \sin x \cdot (1 - \cos^2 x) \, dx$

let $u = \cos x$

$du = -\sin x \, dx$

$dx = \dfrac{du}{-\sin x}$

$= \int \cancel{\sin x} \cdot (1 - u^2) \cdot \dfrac{du}{-\cancel{\sin x}}$

$= -\int 1 - u^2 \, du$

$= -\left(u - \dfrac{u^3}{3}\right) + c$

$= -\left(\cos x - \dfrac{\cos^3 x}{3}\right) + c$

Method of Integral

1- Substitution Rule Method التكامل بالتعويض

$$\int f(x)\, dx$$

خطوات الحل :-

١- نختار فرض مناسب يساوي y

٢- نشتق y لنحصل على dx بدلالة dy

٣- نحول التكامل من x الى y

٤- نجد قيمة التكامل الجديد بدلالة y

٥- بعد حساب التكامل نرجع y لا صلها x

Ex: $\int x \sqrt{x^2 + 7}\, dx$

$$y = x^2 + 7 \longrightarrow \frac{dy}{dx} = 2x \longrightarrow \boxed{dx = \frac{dy}{2x}}$$

$$x^2 = \sqrt{y - 7}$$

$$= \int \cancel{x} \sqrt{y}\, \frac{dy}{2\cancel{x}}$$

$$= \frac{1}{2} \int \sqrt{y}\, dy$$

$$= \frac{1}{2} \int y^{\frac{1}{2}}\, dy$$

$$= \frac{1}{2} \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{\cancel{2}} y^{3/2} \cdot \frac{\cancel{2}}{3} + c$$

$$= \frac{1}{3} y^{3/2} + c$$

$$= \frac{1}{3} (x^2 + 7)^{3/2} + c$$

$Ex: \int x\sqrt{x-3}\,dx$

$$= \int x\,(x-3)^{1/2}\,dx$$

$y = x - 3 \longrightarrow x = y + 3$

$$\frac{dy}{dx} = 1$$

$\therefore dy = dx$

$$= \int x \cdot \sqrt{y}\,dy$$

$$= \int (y+3)\sqrt{y}\,dy$$

$$= \int y^{3/2} + 3y^{1/2}\,dy$$

$$= y^{5/2} \cdot \frac{2}{5} + \cancel{3}y^{\frac{3}{2}} \cdot \frac{2}{\cancel{3}} + C$$

$$= y^{5/2} \cdot \frac{2}{5} + 2y^{3/2} + C$$

$$= (x-3)^{\frac{5}{2}} \cdot \frac{2}{5} + 2(x-3)^{3/2} + C$$

$$\int_0^1 \frac{\tan^{-1}(x)}{1+x^2}\,dx$$

$$y = \tan^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$dx = \frac{dy}{\frac{1}{1+x^2}} \longrightarrow \boxed{dx = (1+x^2)\,dy}$$

$$= \int_0^1 \frac{y}{\cancel{1+x^2}} \cdot \cancel{(1+x^2)}\,dy$$

$$= \int_0^1 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^1 \Longrightarrow \left[\frac{(\tan^{-1}x)^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left[\tan^{-1}(1)^2 - \tan^{-1}(0)^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0\right]$$

$$= \frac{1}{2} \cdot \frac{\pi^2}{16} \Longrightarrow = \frac{\pi^2}{32}$$

H.W

1- $\int (3x-4)^5\,dx$

2- $\int x^3 \sqrt{1-x^4}\,dx$

3- $\int 2x\sqrt{1+x^2}\,dx$

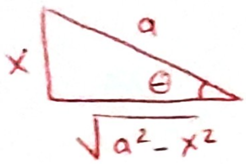
Trigonometric Subsitution

$a - x^2 \to \sin$

$x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

$\sin\theta = \frac{x}{a}$

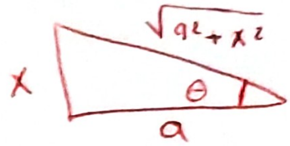


$a^2 + x^2 \to \tan\theta$

$x = a\tan\theta$

$dx = a\sec^2\theta \, d\theta$

$\tan\theta = \frac{x}{a}$

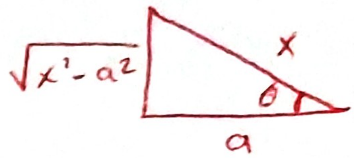


$x^2 - a^2 \to \sec$

$x = a\sec\theta$

$dx = a\sec\theta \tan\theta \, d\theta$

$\sec\theta = \frac{x}{a}$



Ex: $\int \sqrt{4 - x^2} \, d\theta$

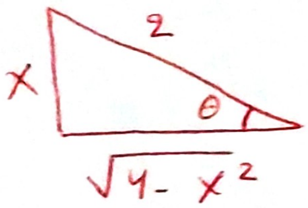
$x = a\sin\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$\sin\theta = \frac{x}{2}$

$\theta = \sin^{-1}\frac{x}{2}$



$$= \int \sqrt{4-(2\sin\theta)^2} \cdot 2\cos\theta \, d\theta$$

$$= \int \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta \, d\theta$$

$$= \int 2\sqrt{1-\sin^2\theta} \cdot 2 \cos\theta \, d\theta$$

$$= \int 4\sqrt{\cos^2\theta} \cdot \cos\theta \, d\theta$$

$$= 4\int \cos^2\theta \, d\theta$$

$$= 4\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= 4\cdot\frac{1}{2}\int 1+\cos 2\theta \, d\theta$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$* \sin 2\theta = 2\sin\theta \cos\theta$

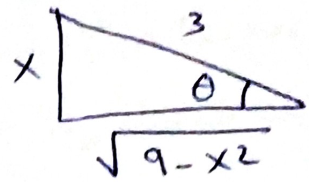
$$= 2\left[\sin^{-1}\frac{x}{2} + \frac{\cancel{2}\sin\theta \cdot \cos\theta}{\cancel{2}}\right] + C$$

$$= 2\left[\sin^{-1}\frac{x}{2} + \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} + C\right]$$

Ex: $\int \frac{dx}{\left(\sqrt{9-x^2}\right)^3}$

ميصير ارفعه لفوق لان مو

بدون التكعيب تطلع اله $\sin^{-1}$



$x = 3\sin\theta$

$dx = 3\cos\theta \, d\theta$

$\sin\theta = \frac{x}{3}$

$\boxed{\theta = \sin^{-1}\frac{x}{3}}$

$= \int \frac{3\cos\theta \, d\theta}{\left(\sqrt{9-(3\sin\theta)^2}\right)^3}$

$= \int \frac{3\cos\theta \, d\theta}{\left(\sqrt{9-9\sin^2\theta}\right)^3}$

$= \int \frac{3\cos\theta \, d\theta}{3\left(\sqrt{1-\sin^2\theta}\right)^3}$

$= \int \frac{3\cos\theta \, d\theta}{\left(3\sqrt{\cos^2\theta}\right)^3}$

$= \int \frac{\cancel{3}\,\cancel{\cos\theta}\, d\theta}{\underset{9}{\cancel{27}}\cos^{\cancel{3}\,2}\theta}$

$= \int \frac{d\theta}{9\cos^2\theta}$

$= \frac{1}{9}\int \frac{1}{\cos^2\theta}\, d\theta$

$= \frac{1}{9}\int \sec^2\theta \, d\theta$

$= \frac{1}{9}\tan\theta + c$

$=$

$= \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + c$

Ex: $\displaystyle\int \frac{\sqrt{x^2-16}}{x^2}\,dx$

$x = 4\sec\theta$

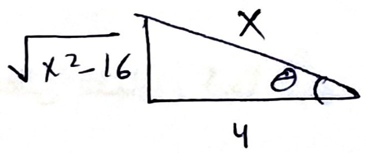
$dx = 4\sec\theta\tan\theta\,d\theta$

$\sec\theta = \dfrac{x}{4}$

$$= \int \frac{\sqrt{16\sec^2\theta-16}}{16\sec^2\theta}\cdot 4\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{4\sqrt{\sec^2\theta-1}}{\cancel{16}_{\,4}\sec^2\theta}\cdot \cancel{4}^{\,1}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\cancel{4}\,\overbrace{\sqrt{\sec^2\theta-1}}^{\tan^2\theta}}{\cancel{4}\sec\theta}\cdot\tan\theta\,d\theta$$



افرض

$\sec^2\theta-1$

$$= \int \frac{\tan^3\theta\,d\theta}{\sec\theta}$$

$$= \int \frac{\sec^2\theta-1}{\sec\theta}\,d\theta$$

$$= \int \frac{\sec^{\cancel{2}}\theta}{\cancel{\sec\theta}} - \frac{1}{\sec\theta}\,d\theta$$

$$= \int \sec\theta - \cos\theta\,d\theta$$

$$= \ln(\sec\theta+\tan\theta) - \sin\theta + c$$

$$= \ln\left(\frac{x}{4} + \frac{\sqrt{x^2-16}}{4} - \frac{\sqrt{x^2-16}}{x}\right) + c$$

by parts Method طريقة التكامل بالاجزاء

$$\int u\,dv = u \cdot v - \int v \cdot du$$

Ex: $\int \ln(x)\,dx$

let $u = \ln(x) \Rightarrow du = \frac{1}{x}$

$dv = dx$

$v = x$

$\therefore \int \ln(x) = \ln(x) \cdot x - \int \cancel{x}\left(\frac{1}{\cancel{x}}\right) dx$

$= \ln(x) \cdot x - \int dx$

$= \ln(x) \cdot x - x + C$

Ex: $\int x \cos x\,dx$

let $u = x$ $\quad$ $dv = \cos x$

$du = 1\,dx$ $\quad$ $v = \sin x$

$\therefore \int x \cos x\,dx = u \cdot v - \int du \cdot v$

$= x \cdot \sin x - \int \sin x \cdot 1\,dx$

$= x \sin x + \cos x + C$

Ex: $\int \tan^{-1}(x)\, dx$

$u = \tan^{-1}(x)$, $dv = dx$

$du = \frac{1}{1+x^2}$, $v = x$

$= \int \tan^{-1}(x) = u.v - \int v.du$

$= x\tan^{-1}(x) - \int x \cdot \frac{1}{1+x^2}\, dx$

$= x \tan^{-1}(x) - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx$

$= x \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + C$

Ex: $\int_0^1 8x\, e^{2x}\, dx$

let $u = 8x$, $dv = e^{2x}$

$du = 8\, dx$ $\quad$ $v = \frac{e^{2x}}{2}$

$= \int_0^1 8x\, e^{2x}\, dx = 8x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{\cancel{2}_{1}} * \overset{4}{8}\, dx$

$= 8x \frac{e^{2x}}{2} - \int 4 e^{2x}\, dx$

$= 4x\, e^{2x} - \int 4 e^{2x}\, dx$

$= 4x\, e^{2x} - \left[\frac{4 e^{2x}}{2} \right]_0^1$

Scanned with ACE Scanner

$$= \left[4(1)\,e^{2} - 2\,e^{2(1)}\right] - \left[0 - 2e^{0}\right]$$

$$= 4e^{2} - 2e^{2} + 2 \qquad e^{0} = 1$$

$$= 2e^{2} + 2$$

Ex: $\int x^3 \sin x \, dx$

u		dv
x^3		$\sin x$
$3x^2$	$+$	$-\cos x$
$6x$	$-$	$-\sin x$
6	$+$	$\cos x$
0	$-$	$\sin x$

$$\int x^3 \sin x \, dx$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Ex: $\int (4x+5)^2 \, e^{x} \, dx$

u		dv
$(4x+5)^2$		e^{x}
$2(4x+5)(4)$	$+$	e^{x}
32	$-$	e^{x}
0	$+$	e^{x}

$$= (4x+5)^2 e^{x} - 8(4x+5)e^{x} + 32e^{x} + C$$

H.w

1- $\int \frac{\ln(x)}{x^2}\,dx$

2- $\int x^2 \ln(x)\,dx$

3- $\int \ln(2x+1)$

Partial fraction: الكسور الجزئيه

$$\int \frac{Poly_1}{Poly_2}\,dx$$

خطوات الحل:

١- نحلل المقام الى عوامله الاوليه

٢- نختار الكسور الجزئيه المناسبه حسب عوامل المقام

٣- نوحد المقامات

٤- نجد قيمة الثوابت المجهولة

٥- نعوض قيمة الثوابت في الكسور الجزئيه ونكاملها

Ex:

* $ax+b = \dfrac{A}{ax+b}$

* $ax^2+bx+c = \dfrac{Ax+B}{ax^2+bx+c}$

* $(ax+b)^n = \dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2} + \cdots\cdots + \dfrac{\quad}{(ax+b)^n}$

* $(ax^2+bx+c)^n = \dfrac{Ax+B}{ax^2+bx+c} + \dfrac{Cx+D}{(ax^2+bx+c)^2} + \cdots\cdots + \dfrac{\quad}{(ax^2+bx+c)^n}$

$X: \int \frac{x-4}{x^2+x-2}\,dx$

$$= \int \frac{x-4}{\underbrace{x^2+x-2}_{(x+2)(x-1)}}\,dx = \int \frac{A}{x+2} + \frac{B}{x-1}\,dx \;\cdots\cdots *$$

$$= \int \frac{x-4}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

∵ المقامات متساوية

∴ يتساوى البسط

$$(x-4) = A(x-1) + B(x+2)$$

let $x=1 \Rightarrow 1-4 = B(1+2)$

$$-3 = 3B \Rightarrow \boxed{B=-1}$$

let $x=-2 \Rightarrow -2-4 = A(-2-1)$

$$-6 = -3A$$

$$\boxed{A=2}$$

$$\int \frac{2}{x+2} + \frac{-1}{x-1}\,dx$$

$$= 2\ln|x+2| - \ln|x-1| + C$$

$$\int \frac{x-1}{x^3+x^2}$$

$$= \int \frac{x-1}{x^2(x+1)}$$

$$= \int \frac{x-1}{x^2(x+1)} = \int \frac{A}{x} + \int \frac{B}{x^2} + \int \frac{C}{x+1} \quad \text{-----} \quad *$$

$$\frac{x-1}{x^2(x+1)} = \frac{A(x+1)x^2 + B(x+1) + x^2 C}{x^2(x+1)}$$

$$(x-1) = A(x+1)x^2 + B(x+1) + Cx^2$$

let

$x = -1 \Rightarrow -2 = 0 + 0 + C \Rightarrow \boxed{C = -2}$

$x = 0 \Rightarrow -1 = A + B \Rightarrow \boxed{B = -1}$

$x = 1 \Rightarrow 0 = 2A + 2B + C$

$0 = 2A + 2(-1) - 2$

$0 = 2A - 2 - 2$

$0 = 2A - 4$

$2A = 4 \Rightarrow \boxed{A = 2}$

$$\int \frac{2}{x} + \int \frac{-1}{x^2} + \int \frac{-2}{x+1} \, dx$$

$$= \int \frac{2}{x} - \int x^{-2} + \int \frac{-2}{x+1} \, dx$$

$$= 2 \ln|x| - \frac{x^{-1}}{-1} - 2 \ln|x+1| + C$$