عنوان المحاضرة:

Functions of Limits, Algebraic Limits, Trigonometric Limits, infinity as Limits دالة الغايات، غايات جبرية، غايات الدوال المثلثية، غايات لامحددة

Function of Limits : دالة الغايات أو دالة النهايات

- Limits describle how a function behaves near a point.

In other words;

- The limits are defined as the value that the Function approaches as it goes to an x value

تعرف النهايات بأنها قيمة الدالة عندما تقترب من قيمة المتغير الغير معتمد x.

What is limit in calculus in real life :

ماهو النهاية في الحساب في الحياة الحقيقية

The real life limits are used any time. Where measuring the temperature of an ice cube imearsed/sunked (مغمور / مغطس) in a warm glass of water is a limit, this is one example. other example is measuring of an electric is a limit, magnetic or gravitational fields are all limits. All are reaches or approaches a steady solution.

- A Judgement day is the approaching day that the whole live is going to!!

EX1: Find $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2}$

Solution

توجد عدة طرق لحساب قيمة الدالة عندما $x \to 2$، وأولها بالتعويض المباشر وملاحظة الحل النهائي فيما اذا كان محلل أم لا.

(1) direct substitution :

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0} = \text{undefined}$$

- هنا لا يوجد حل، ولهذا يجب أن نفكر بطريقة للحل أخرى.

② Some times we can plug in a value of x that is close to 2, if not exactly 2.

- let take $x = 2.1$

$\therefore f(2.1) = \frac{(2.1)^2 - 4}{2.1 - 2} = \frac{0.41}{0.1} = 4.1$

- Let take another value that is closer to 2.
- let take $x = 2.01$

$f(2.01) = \frac{(2.01)^2 - 4}{2.01 - 2} = 4.01$

Note that as we get closer & closer to 2 the limit approaches 4

So, we can say that $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

* This technique works to any limit, as long as we plug in a number that's very close to whatever the limit approach to a number is, but not exactly the number & if the limit exists is going to converge to a certain value !!

③ However, there are another techniques to get the answer,

lets solve the same example by,

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x \to 2} (x+2) = 2+2 = \boxed{4}$$

Ex (2): $\lim_{x \to 5} (x^2+2x-4)$

Solution

Here we have no fraction that may equals to zero, so, we can always use the direct substitution

$$\lim_{x \to 5} (x^2+2x-4) = 5^2+2(5)-4 = 35-4 = \boxed{31}$$

Ex (3) $\lim_{x \to 3} \frac{x^3-27}{x-3}$

Solution

Here we can't use the direct substitution as the fraction will be "$\frac{0}{0}$ = undefined solution" So, in such a case we may think about using a difference of cubes, like;

$$A^3 - B^3 = (A-B)(A^2+AB+B^2)$$

Compare this expression with what we have in the Numerator "البسط"

$A^3 = x^3$

$B^3 = 27$

$$\therefore \lim_{x \to 3} \frac{x^3-27}{x-3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3} (x^2+3x+9)$$

Here we can use a direct substitution

$$\lim_{x \to 3} (x^2+3x+9) = 3^2+3(3)+9 = \boxed{27}$$

Ex (4) $\lim_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$

Solution

In such problems (having a complex Fraction), we need to multiply the numerator & the denominator by a common denominator of the two fraction $\frac{1}{x}$ & $\frac{1}{3}$ which is '$3x$'

$$\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} * \frac{3x}{3x} = \lim_{x\to 3} \frac{3-x}{3x(x-3)}$$

$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}}$$

$$= \lim_{x\to 3} \frac{-1}{3x} = \frac{-1}{3(3)} = \boxed{\frac{-1}{9}}$$

Ex (5) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

Solution

In such case, we need to multiply the top & the bottom by the numerator's Conjugate

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} * \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)} = \lim_{x\to 9} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3}$$

$$= \boxed{\frac{1}{6}}$$

Ex (6) $\lim_{x \to 4} \dfrac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4}$

Solution

In such case, we need to multiply the top & the bottom of the eqs not just by the common denominator but also by the Conjugate !!

- lets start with a common denominator, which is "$2\sqrt{x}$"

$$\lim_{x \to 4} \frac{\left(\frac{1}{\sqrt{x}} - \frac{1}{2}\right)}{(x-4)} * \frac{2\sqrt{x}}{2\sqrt{x}} = \lim_{x \to 4} \frac{2 - \sqrt{x}}{2\sqrt{x}(x-4)}$$

- Now we need to multiply the top & the bottom of the eqs by the conjugate $(2+\sqrt{x})$

$$\therefore \lim_{x \to 4} \frac{(2-\sqrt{x})}{2\sqrt{x}(x-4)} * \frac{(2+\sqrt{x})}{(2+\sqrt{x})} = \lim_{x \to 4} \frac{4-x}{2\sqrt{x}(x-4)(2+\sqrt{x})}$$

$$= \lim_{x \to 4} \frac{-\cancel{(x-4)}}{2\sqrt{x}\cancel{(x-4)}(2+\sqrt{x})}$$

$$= \lim_{x \to 4} \frac{-1}{2\sqrt{x}(2+\sqrt{x})} = \frac{-1}{2\sqrt{4}(2+\sqrt{4})}$$

$$= \boxed{\frac{-1}{16}}$$

Algebric Limits:- النهايات الجبرية

The limits of numbers & varsables togethers Known as algebra of limits-

Properties الخواص

1- $\lim\limits_{x\to a} c = c$ "The limit of a constant is equal to the constant"

2- $\lim\limits_{x\to a} x = a$ "The limit of x as x approaches a is equal to a"

3- $\lim\limits_{x\to a} [f(x) \mp g(x)] = \lim\limits_{x\to a} f(x) \mp \lim\limits_{x\to a} g(x)$

"The limit of a sum is the sum of the limits-" (likewise for differences)

4- $\lim\limits_{x\to a} [c\, f(x)] = c \lim\limits_{x\to a} f(x)$

"The limit of a constant times a Fun is the constant times the limit of the fun.

5- $\lim\limits_{x\to a} [f(x)\, g(x)] = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$

"The limit of a product is the product of the limits"

6- $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, $\lim\limits_{x\to a} g(x) \neq 0$

"The limit of a quotient is the quotient of the limits, the denominator $\neq 0$."

7- $\lim\limits_{x\to a} [f(x)]^n = \left(\lim\limits_{x\to a} f(x)\right)^n$, n = rational number رقم نسبي (كسري)

مثل 4.12 or (-2.6), 8.56

"The limit of a power is the power of the limit, n = rational number".

8- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, if the root on the right side exists.

"The limit of a root is the root of the limit provided that the root exists"

Ex(7) Evaluate $\lim_{x \to 1} \dfrac{x^2+x-2}{x^2-x}$

Solution

The direct substitution on x by 1 is not suitable idea as it gives $\frac{0}{0}$ = undefined qu.

$$\therefore \lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}}$$

$$= \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = \boxed{3}$$

Trigonometric limits :- نهايات الدوال المثلثية

Theorm ①

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians})$$

Theorm ② Sandwich Theorm

IF $g(x) \le f(x) \le h(x)$ for all x in open interval containing c, then,

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

Then, $\lim_{x \to c} f(x) = L$

Ex ⑧

IF $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ for all $x \ne 0$

Find $\lim_{x \to 0} u(x)$, no matter how complicated u is

Solution,

Since, $\lim_{x \to 0} 1 - \frac{x^2}{4} = \boxed{1}$

And, $\lim_{x \to 0} 1 + \frac{x^2}{2} = \boxed{1}$

The sandwich Theorm implies that $\lim_{x \to 0} u(x) = \boxed{1}$

Ex ⑨

IF $-|\theta| \le \sin\theta \le |\theta|$ For all θ, Find $\lim_{\theta \to 0} \sin\theta$

Solution since, $\lim_{\theta \to 0} (-|\theta|) = \lim_{\theta \to 0} (|\theta|) = 0$

$\therefore \lim_{\theta \to 0} \sin\theta = \boxed{0}$

EX(10)

IF $0 \leq 1-\cos\theta \leq |\theta|$ For all θ, Find $\lim_{\theta \to 0} (1-\cos\theta)$

Sol.

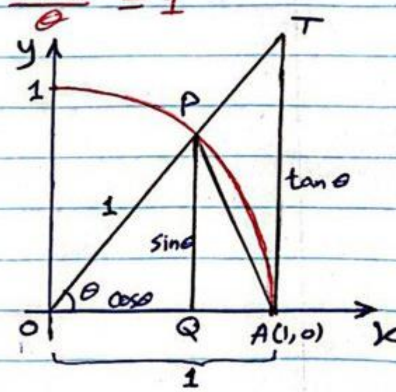
Since, $\lim_{\theta \to 0} (0) = \lim_{\theta \to 0} |\theta| = 0$

Then, $\lim_{\theta \to 0} (1-\cos\theta) = 0 \Longrightarrow \lim_{\theta \to 0} 1 - \lim_{\theta \to 0} \cos\theta = 0$

$\therefore 1 - \lim_{\theta \to 0} \cos\theta = 0 \longrightarrow \lim_{\theta \to 0} \cos\theta = \boxed{1}$

EX(11) proof that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Sol.



The plan is to show that the right-hand & the left hand limits are both 1. Then We will Know that the two-sided limit is 1 as Well.

Area Δ OAP < area sector OAP < area Δ OAT

We can express these areas in terms of θ as Follows,

- Area $\Delta OAP = \frac{1}{2}$ base $\times$ height $= \frac{1}{2}(1)(\sin\theta) = \boxed{\frac{1}{2}\sin\theta}$
- Area sector $OAP = \frac{1}{2} r^2 \theta = \frac{1}{2}(1)^2\theta = \boxed{\frac{\theta}{2}}$
- Area $\Delta OAT = \frac{1}{2}$ base $\times$ height $= \frac{1}{2}(1)(\tan\theta) = \boxed{\frac{1}{2}\tan\theta}$

Thus,

$$\left(\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta\right) \times \frac{2}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Take reverse,

$$\left(1 < \frac{\sin\theta}{\theta} < \cos\theta\right) \lim_{\theta\to 0}$$

$$\lim_{\theta\to 0} 1 < \lim_{\theta\to 0} \frac{\sin\theta}{\theta} < \lim_{\theta\to 0} \cos\theta \quad ;$$

$$1 < \lim_{\theta\to 0} \frac{\sin\theta}{\theta} < 1$$

From the sandwich Therm,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

o.k

Ex (12)

Show that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

Sol.

Using the half-angle Formula

$$\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right)$$

$$\therefore \lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{\cancel{1} - 2\sin^2\left(\frac{h}{2}\right) - \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$$

put $\boxed{h = 2\theta} \longrightarrow h\to 0 = \theta\to 0$

$$= -\lim_{\theta\to 0} \frac{\cancel{2}\sin\theta * \sin\theta}{\cancel{2}\theta}$$

$$= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} * \sin\theta = -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} * \lim_{\theta\to 0} \sin\theta$$

1, 0

$$= -(1)(0) = \boxed{0}$$

o.k

EX (13) Show that $\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Sol.

$$\lim_{x\to 0} \frac{\sin 2x}{5x} * \frac{2/5}{2/5} = \lim_{x\to 0} \frac{(2/5)*\sin 2x}{(2/\cancel{5})*\cancel{5}x}$$

$$= \lim_{x\to 0} \frac{2}{5} * \frac{\sin 2x}{2x}$$

$$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x} \quad , \text{put } 2x = \theta$$

$$= \frac{2}{5}(1) = \boxed{\frac{2}{5}} \quad \text{o.k}$$

Infinity as limits :- غايات/نهايات لا محدودة

Here, we investigate the behavior of a Fun when the magnitude of the independent variables increasingly large, or $x \to \mp\infty$

will take examples to deal with all the cases associated with Limit at Infinity

- Ex (14) Find $\lim_{x \to \infty} x^2$

Sol.

$$\lim_{x \to \infty} x^2 = (\infty)^2 = \boxed{+\infty}$$

- Ex (15) Find $\lim_{x \to -\infty} x^2$

Sol.

$$\lim_{x \to -\infty} x^2 = (-\infty)^2 = \boxed{+\infty}$$

- Ex (16) Find $\lim_{x \to \infty} x^3$

Sol.

$$\lim_{x \to \infty} x^3 = (\infty)^3 = \boxed{+\infty}$$

- Ex (17) Find $\lim_{x \to -\infty} x^3$

Sol.

$$\lim_{x \to -\infty} x^3 = (-\infty)^3 = \boxed{-\infty}$$

Ex (18) Find $\lim_{x\to-\infty}(5+x-x^3)$

Sol.

In such problems, we can ignore the small or insignificant parts of the polynomial equation & keep the heavy parts only.

هنا، كاستراتيجية للحل، يمكننا أن نهمل الحدود الصغيرة ونبقي الكبيرة فقط.

(Equivalent)

$$\lim_{x\to-\infty}(\cancel{5}+\cancel{x}-x^3) \approx \lim_{x\to-\infty}(-x^3) = -(-\infty)^3$$

$$= -(-\infty) = \boxed{+\infty}$$

Ex (19) Find $\lim_{x\to-\infty}(3x^3-5x^4)$

Sol.

$$\lim_{x\to-\infty}(\cancel{3x^3}-5x^4) \approx \lim_{x\to-\infty}(-5x^4) = -5(-\infty)^4$$

$$= -5(+\infty)$$

$$= \boxed{-\infty}$$

Ex (20) Find $\lim_{x\to\infty}\left(\frac{1}{x}\right)$ "rational Fn"

Sol.

$$\lim_{x\to\infty}\left(\frac{1}{x}\right) = \frac{1}{\infty} = \boxed{0}$$

Notes

$\frac{1}{10} = 0.1$

$\frac{1}{100} = 0.01$

$\frac{1}{1000} = 0.001$

$\vdots \qquad \vdots$

$\frac{1}{\infty} = 0.0000000\ldots = 0$

Ex (21) Find $\lim_{x\to-\infty}\frac{1}{x^2}$

Sol.

Similar to the Ex (20)

$$\lim_{x\to-\infty}\frac{1}{x^2} = \frac{1}{(-\infty)^2} = \frac{1}{\infty} = \boxed{0}$$

Ex (22) Find $\lim_{x\to\infty} \frac{5x+2}{7x-x^2}$

Sol.

In such problems or rational Fns, we look to the powers (degree)

- For the numerator $\longrightarrow 5x^{①}+2$
For the denominator $\longrightarrow 7x-x^{②}$] ← we have a bottom heavy Fn

Note) Any rational Fn with bottom heavy of the limit is going to $(\mp\infty)$, then its going to be Zero

ملاحظة
- في حالة المسائل ذات الدوال الكسرية ، ننظر الى درجة معادلة البسط والمقام فاذا كانت درجة أو اعلى أس للمقام اكبر من اعلى أس للبسط. وكانت الغاية متجه الى $(\mp\infty)$ فهنا الجواب يكون صفراً.
واثباتها كالتالي :

$$\lim_{x\to\infty} \frac{(5x+2)}{(7x-x^2)} * \left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty} \frac{5/x + 2/x^2}{7/x - 1}$$

$$= \frac{5/\infty + 2/(\infty)^2}{7/\infty - 1} = \frac{0+0}{0-1} = \frac{0}{-1} = \boxed{0}$$

Ex (23) Find $\lim_{x\to\infty} \frac{6x^{②}-4x}{3x^{②}+5}$

Sol.

Here the degrees are same, so the answer will be simply just divide the coefficients

$$\frac{6}{3} = \boxed{2}$$

ملاحظة : في حالة تساوي درجة البسط والمقام فالجواب يكون بقسمة معاملي المتغيرين ذات الاس الاعلى المتساوي فقط ، كما في المثال التالي :-

$$\lim_{x\to\infty} \frac{(6x^2-4x)}{(3x^2+5)} \times \left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty} \frac{6-\cancel{4/x}^{\,0}}{3+\cancel{5/x^2}^{\,0}} = \frac{6-0}{3-0}$$

$$= \frac{6}{3} = \boxed{2}$$

Ex (24) Find $\lim\limits_{x \to \infty} \dfrac{5x+6x^2}{3x-8}$

Sol-

$$\lim_{x \to \infty} \frac{(5x+6x^2)}{(3x-8)} * \frac{1/x}{1/x} = \lim_{x \to \infty} \frac{5+6x}{3-8/x}$$

$$= \frac{5+6(\infty)}{3-\cancel{8/\infty}^{\,0}} = \frac{5+\infty}{3-0} = \boxed{+\infty}$$

Ex (25) Find $\lim\limits_{x \to -\infty} \dfrac{5+2x-3x^3}{4x^2+9x-7}$

Sol-

كحل سريع يمكننا اهمال الحدود الصغيرة ونبقيا الحدود ذات الاس العالية
فقط كما في التالي

$$\lim_{x \to -\infty} \frac{\cancel{5}+\cancel{2x}-3x^3}{4x^2+\cancel{9x}\cancel{-7}} \approx \lim_{x \to -\infty} \frac{-3x^{\cancel{3}\,1}}{4\cancel{x^2}}$$

$$= \lim_{x \to -\infty} \frac{-3x}{4} = \frac{-3(-\infty)}{4}$$

$$= \frac{3}{4}(\infty) = \boxed{+\infty}$$

HW # 4

① Proof that $\lim\limits_{t \to 0} \dfrac{\tan t \sec 2t}{3t} = \dfrac{1}{3}$

② Find $\lim\limits_{x \to -\infty} \dfrac{5x-7x^3}{2x^2+14x^3-9}$

نهاية محاضرة " Functions of Limits, Algebraic Limits, Trigonometric Limits, infinity as Limits دالة الغايات، غايات جبرية، غايات الدوال المثلثية، غايات لامحددة"