Stability Analysis

Stability is a very important characteristic of the transient performance of a system as every system has to pass through a transient stage for a small period befor reaching steady state.

stability in system indicates that small changes in the system input in initial conditions or in system parameter, do not result in large changes in the system output.

Conditions of Stability

Consider a system with characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

① all the coefficients of the characteristic eq. should have same sign.

② all powers of s must present in descending order or there should be no missing term.

if any condition or both are not satisfied the system will be unstable

The Routh - Hurwitz Criterion

This criterion is based on ordering the coefficients of th c/e into an array called the Routh's Array

$$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

$$\begin{array}{cccc} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ b_1 & b_2 & b_3 & \cdots \\ c_1 & c_2 & c_3 & \cdots \end{array}$$

where $b_1 = \dfrac{a_1 a_2 - a_0 a_3}{a_1}$, $b_2 = \dfrac{a_1 a_4 - a_0 a_5}{a_1}$, $b_3 = \dfrac{a_1 a_6 - a_0 a_7}{a_1}$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{a_5 b_1 - a_1 b_3}{b_1}$$

Ex: 1 Check the stability of the system whose characteristic equation is given by $s^3 + 5s^2 + 10s + 3 = 0$

s^3	1	10
s^2	5	3
s^1	9.4	0
s^0	3	

$$\frac{5 \times 10 - 1 \times 3}{5} = \frac{47}{5}, \quad \frac{5 \times 0 - 1 \times 0}{5} = 0$$

$$\frac{9.4 \times 3 - 5 \times 0}{9.4} = 3.$$

the first column there is no change of sign, therefore the system is stable.

Ex: 2 نفس المنطوق السابق

$$s^3 + 2s^2 + 3s + 10 = 0$$

s^3	1	3
s^2	2	10
s^1	-2	0
s^0	10	

there are two change of sign in the first column (+2 to -2 , and from -2 to 10) therefore the system in unstable.

Ex: 3 ch./eq. $s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1 = 0$

s^5	1	4	3
s^4	2	8	1
s^3	0	2.5	0
s^2	∞	∞	
s^1			
s^0			

$\leftarrow$ Routh's Array Break down.

replace 0 by ϵ (small positive number) and complet.

s^5	1	4	3
s^4	2	8	1
s^3	ϵ	2.5	0
s^2	$\frac{8\epsilon-5}{\epsilon}$	1	0
s^1	$\frac{2.5\left(\frac{8\epsilon-5}{\epsilon}\right)-\epsilon}{\frac{8\epsilon-5}{\epsilon}}$		0
s^0	1		

$$\lim_{\epsilon \to 0} \frac{8\epsilon-5}{\epsilon} = 8-\frac{5}{\epsilon} = 8-\infty = -\infty$$

$$\lim_{\epsilon \to 0} \frac{2.5\left(\frac{8\epsilon-5}{\epsilon}\right)-\epsilon}{\frac{8\epsilon-5}{\epsilon}} = 2.5$$

There are two change of sign ; the system is unstable.

OR by replaced $s \longrightarrow \frac{1}{z}$

$$s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1 = 0$$

$$\frac{1}{z^5} + \frac{2}{z^4} + \frac{4}{z^3} + \frac{8}{z^2} + \frac{3}{z} + 1 = 0 \qquad * z^5$$

$$1 + 2z^1 + 4z^2 + 8z^3 + 3z^4 + z^5 = 0$$

$$z^5 + 3z^4 + 8z^3 + 4z^2 + 2z + 1 = 0$$

z^5	1	8	2
z^4	3	4	1
z^3	6.67	1.67	0
z^2	3.25	1	0
z^1	-0.382	0	0
z^0	1	0	0

two change in sign. the system is unstable.

- Auxiliary Equation

The polynomial whose coefficients are the elements of the row just above the row of zeros in the Routh Array is called Auxiliary equation.

This have ① symmetrically located in the s-plane

② the order is always even.

Ex: $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

s^6	1	8	20	16
s^5	2	12	16	0
s^5	1	6	8	
s^4	2	12	16	
s^4	1	6	8	
s^3	0	0		

divide all coeffi. of s^5 row by 2

→ divide all cof. of s^4 raw by 2

s^3 row: 0 0 ⟶ Routh's Array Break down.

نلاحظ بعد التقسيم للصف الرابع s^4 كانت النتيجة في الصف الثالث s^3 تساوي صفراً

لذا نعمل) اخذ معادلة الصف الرابع s^4 —— $A(s) = s^4 + 6s^2 + 8$

ثم نضع هذا الصف بالنظام وكالاتي $\dfrac{dA(s)}{ds} = 4s^3 + 12s$

s^6	1	8	20	16
s^5	1	6	8	
s^4	1	6	8	
s^3	4	12		
s^2	3	8		
s^1	$\frac{4}{3}$			
s^0	8			

→ بعد التقسيم $\frac{}{2}$

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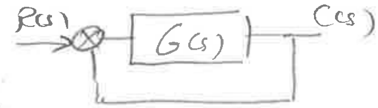
The system is stable.

Ex: 1 / For a unity feed back control system having open-loop T-F

$$G(s) = \frac{K}{s(1+0.6s)(1+0.4s)}$$, Determine the range of K. and frequency of sustained oscillations.



Sol

The characteristic equation of the system is given by

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(1+0.6s)(1+0.4s)} = 0$$

$$s(1+0.6s)(1+0.4s) + K = 0$$

$$s(1+s+0.24s^2) + K = 0$$

$$0.24s^3 + s^2 + s + K = 0$$

Routh's Array becomes $\Longrightarrow$

$$\begin{array}{c|cc} s^3 & 0.24 & 1 \\ s^2 & 1 & K \\ s^1 & 1-0.24K & 0 \\ s^0 & K & \end{array}$$

the system will be stable if no change of sign occurs in first column.

$$\therefore K > 0 \Longrightarrow 1 - 0.24K > 0 \Longrightarrow K < \frac{1}{0.24}$$

$$K < 4.167$$

or the range of $K \Longrightarrow \quad 0 < K < 4.167$

to find the frequency of osillation, take the auxiliary equation from Routh's array $\Longrightarrow A(s) = s^2 + K$

$$s^2 = -K$$

$$s^2 = -4.167$$

$$s = \pm 2.04j$$

$$\therefore s = \omega j$$

$$\omega j = 2.04j \Longrightarrow \omega = 2.04\ r/s.$$

Ex:2/ Determine the stability of a system having characteristic equation

$$2s^4 + 4s^2 + 1 = 0$$

sol

$$2s^4 + 0s^3 + 4s^2 + 0s + 1 = 0$$

$$\begin{array}{c|ccc} s^4 & 2 & 4 & 1 \\ s^3 & 0 & 0 & \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array} \longrightarrow \text{Breakdown.}$$

$$A(s) = 2s^4 + 4s^2 + 1 \implies \frac{dA(s)}{ds} = 8s^3 + 8s$$

$$\begin{array}{c|ccc} s^4 & 2 & 4 & 1 \\ s^3 & 8 & 8 & \\ s^2 & 2 & 1 & \\ s^1 & 4 & 0 & \\ s^0 & 1 & & \end{array}$$

no sign change in first column, the system is stable.

Ex:3/ The open-loop T.F of a feedback control system is given by

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+2s+2)}$$, Find the range of value of K for stability

also determine the stability of system when $K = 12$.

sol ch. eq. $\implies 1 + G(s)H(s) = 0 \implies 1 + \dfrac{K}{s(s+4)(s^2+2s+2)} = 0$

$$s(s+4)(s^2+2s+2) + K = 0$$

$$s^4 + 6s^3 + 10s^2 + 8s + K = 0 \longrightarrow \text{Routh's Array} \implies$$

$$\begin{array}{c|ccc} s^4 & 1 & 10 & K \\ s^3 & 6 & 8 & 0 \\ s^2 & \frac{52}{6} & K & \\ s^1 & 8 - \frac{36}{52}K & 0 & \\ s^0 & K & & \end{array}$$

For stability, first column of Routh's Array should be same sign.

$K > 0$, $8 - \frac{36}{52}K > 0 \Rightarrow K < 11.56$

the range of $K \Rightarrow 0 < K < 11.56$

if $K = 12 \Rightarrow$ Ch. eq. $s^4 + 6s^3 + 10s^2 + 8s + 12 = 0$

$$
\begin{array}{c|ccc}
s^4 & 1 & 10 & 12 \\
s^3 & 6 & 8 & 0 \\
s^2 & \frac{52}{6} & 12 & \\
s^1 & -0.30 & 0 & \\
s^0 & 12 & &
\end{array}
$$

the system is unstable, two change in sign of first column.

H-W

Q1/ Check the stability of system $G(s) = \dfrac{K(s+10)(s+20)}{s^2(s+2)}$

Q2/ Use Routh's criterion to investigate the stability of a unity feedback system $G(s) = \dfrac{K}{s(s+2)(s^2+3)}$

Q3/ The open loop T.F of a certain unity feedback system is

$G(s)H(s) = \dfrac{K(s+1)}{s(s-1)(s+6)}$, Determine the range of K for which the system is stable.

Q4/ For a unity feedback control system with $G(s) = \dfrac{K}{(s+1)^3(s+4)}$, determine the range of K for stability and the frequency of oscillations

Q5/ Investigate the stability of the system having characteristic equation

$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

Q6/ For the unity feedback systems, determine the range of K to ensure stability ① $G(s) = \dfrac{K(s^2+1)}{(s+1)(s+2)}$, ② $G(s) = \dfrac{K(s+6)}{s(s+1)(s+3)}$ ③ $G(s) = \dfrac{K(s+2)(s-2)}{(s^2+3)}$