

2.3 Complex Number

Complex Numbers

Definitions.

$$\begin{aligned} \text{Let } i^2 &= -1. \\ \therefore i &= \sqrt{-1}. \end{aligned}$$

Complex numbers are often denoted by z .

Just as \mathbb{R} is the set of real numbers, \mathbb{C} is the set of complex numbers. If z is a complex number, z is of the form

$$z = x + iy \in \mathbb{C}, \text{ for some } x, y \in \mathbb{R}.$$

e.g. $3 + 4i$ is a complex number.

$$\begin{array}{ccc} z = x + iy & & \\ \uparrow & \swarrow & \\ \text{real part} & & \text{imaginary part.} \end{array}$$

If $z = x + iy$, $x, y \in \mathbb{R}$,

$$\text{the real part of } z = \Re(z) = \text{Re}(z) = x$$

$$\text{the imaginary part of } z = \Im(z) = \text{Im}(z) = y.$$

e.g. $z = 3 + 4i$

$$\Re(z) = 3$$

$$\Im(z) = 4.$$

Example// Express each of the following in terms of i .

1. $\sqrt{-25} \rightarrow \sqrt{25i^2} = 5i$

2. $\sqrt{-12} - \sqrt{-3} \rightarrow \sqrt{12i^2} - \sqrt{3i^2} = 2\sqrt{3}i - \sqrt{3}i = \sqrt{3}i$

3. $-2 + \sqrt{-4} \rightarrow -2 + \sqrt{4i^2} = -2 + 2i$

Operation of complex number

ADDITION OF COMPLEX NUMBERS

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

SUBTRACTION OF COMPLEX NUMBERS

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

MULTIPLICATION OF COMPLEX NUMBERS

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

DIVISION OF COMPLEX NUMBERS

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2} \right) i$$

Example 1. $(2 + 3i) + (4 + i) = 6 + 4i$.

Example 2. $(8 - 3i) - (-2 + 4i) = 10 - 7i$.

Multiplication/Division.

Example 1. $(2 + 3i)(1 + 2i) = 2 + 4i + 3i - 6 = -4 + 7i$

Example 2. $(3 - 2i)(3 + 2i) = 9 - (2i)^2 = 9 + 4 = 13$

∴ when we multiply two complex conjugates, we get a real number.

Example 3. $\frac{2+3i}{1+4i} = \frac{2+3i}{1+4i} \times \frac{1-4i}{1-4i} = \frac{(2+3i)(1-4i)}{(1+4i)(1-4i)} = \frac{2-8i+3i-12i^2}{1-(4i)^2} = \frac{14-5i}{17}$

ABSOLUTE VALUE

The absolute value or modulus of a complex number $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$.

Example: $|-4 + 2i| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$

If $z_1, z_2, z_3, \dots, z_m$ are complex numbers, the following properties hold.

1. $|z_1 z_2| = |z_1| |z_2|$
2. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
3. $|z_1 + z_2| \leq |z_1| + |z_2|$
4. $|z_1 - z_2| \leq |z_1| + |z_2|$

Example//show that $|Z1.Z2| = |Z1|. |Z2|$

Solution//

Let $z_1 = a_1 + b_1i, z_2 = a_2 + b_2i$

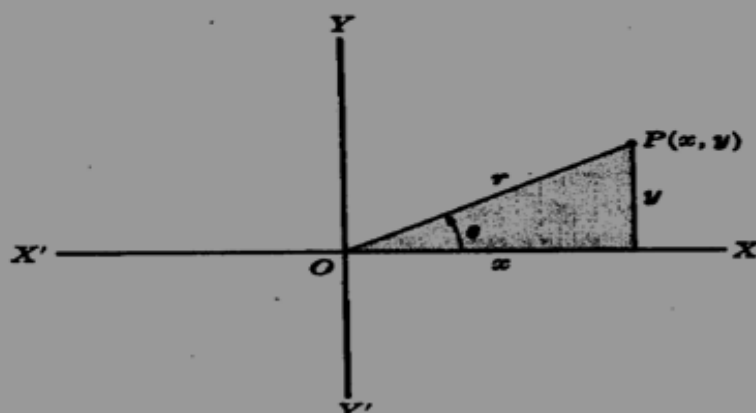
$$\begin{aligned} |(a_1 + b_1i).(a_2 + b_2i)| &= |(a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i| \\ &= \sqrt{(a_1a_2 - b_1b_2)^2 + (b_2a_1 + b_1a_2)^2} \\ &= \sqrt{(a_1^2a_2^2 - 2a_1a_2b_1b_2 + b_1^2b_2^2 + a_1^2b_2^2 + 2a_1a_2b_1b_2 + b_1^2a_2^2)} \\ &= \sqrt{(a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + b_1^2a_2^2)} \end{aligned}$$

While $|Z1|. |Z2| = |a_1 + b_1i|. |a_2 + b_2i|$

$$\begin{aligned} &= \sqrt{a_1^2 + b_1^2} . \sqrt{a_2^2 + b_2^2} \\ &= \sqrt{(a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + b_1^2a_2^2)} \end{aligned}$$

POLAR FORM OF COMPLEX NUMBERS

If P is a point in the complex plane corresponding to the complex number (x, y) or $x + iy$,



where $r = \sqrt{x^2 + y^2} = |x + iy|$ is called the *modulus* or *absolute value* of $z = x + iy$ [denoted by *mod* z or $|z|$]; and θ , called the *amplitude* or *argument* of $z = x + iy$ [denoted by *arg* z], is the angle which line OP makes with the positive x axis.

It follows that

$$z = x + iy = r(\cos \theta + i \sin \theta) \quad (1)$$

which is called the *polar form* of the complex number, and r and θ are called *polar coordinates*. It is sometimes convenient to write the abbreviation *cis* θ for $\cos \theta + i \sin \theta$.

$$x = r \cos \theta, y = r \sin \theta$$

Example// Express the polar form of $3+3i$.

Solution/ $z=3+3i=r[\cos \theta + i \sin \theta]$

$$x=r \cos \theta = 3, y = r \sin \theta = 3$$

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = \pm 3\sqrt{2}$$

$$x=r \cos \theta \rightarrow 3 = 3\sqrt{2} \cos \theta \rightarrow \theta = 45$$

$$z=3+3i=3\sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]$$

RELATIONSHIP BETWEEN EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

$$e^{i\theta} = \cos \theta + i \sin \theta, e^{-i\theta} = \cos \theta - i \sin \theta$$

Eulers theorem :-

$$e^{i\theta} = (\cos \theta + i \sin \theta) * r$$

$$re^{i\theta} = r(\cos \theta + i \sin \theta) \dots\dots 1$$

$$re^{-i\theta} = r(\cos \theta + i \sin \theta) \dots\dots 2$$

Adding 1 and 2

$$re^{i\theta} + re^{-i\theta} = 2r \cos \theta \rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{sub in 1}$$

$$\rightarrow \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Example// show that $\cos^2 \theta + \sin^2 \theta = 1$

Solution //

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos^2 \theta + \sin^2 \theta = \frac{e^{2i\theta}}{4} + e^0 + \frac{e^{-2i\theta}}{4} + \frac{-e^{-2i\theta}}{4} + e^0 + \frac{-e^{-2i\theta}}{4}$$

$$e^0 + e^0 = 1 + 1 = 2$$