



Al-Mustaqbal University
College of Science
Intelligent Medical System Department



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AL MUSTAQBAL UNIVERSITY

كلية العلوم
قسم الانظمة الطبية
الذكائية

Lecture: (9)

Integrals of Triangular Functions

Subject: mathematics

Class: First

Lecturer: Dr. Mustafa Talal

$$\begin{aligned}
 10) \int \frac{\cot^2 \sqrt{x}}{\sqrt{x}} dx &= \int \frac{\csc^2 \sqrt{x} - 1}{\sqrt{x}} dx = 2 \int \frac{\csc^2 \sqrt{x}}{2\sqrt{x}} - \int x^{-1/2} dx \\
 &= 2 \left(-\cot \sqrt{x} \right) - \frac{x^{1/2}}{1/2} + c = -2 \cot \sqrt{x} - 2\sqrt{x} + c
 \end{aligned}$$

5-3- Integrals of inverse trigonometric functions:

The integration formulas for the inverse trigonometric functions are:

$$16) \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c = -\cos^{-1} \frac{u}{a} + c \quad ; \quad \forall u^2 < a^2$$

$$17) \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c = -\frac{1}{a} \cot^{-1} \frac{u}{a} + c$$

$$18) \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + c = -\frac{1}{a} \csc^{-1} \left| \frac{u}{a} \right| + c \quad ; \quad \forall u^2 > a^2$$

EX-3 Evaluate the following integrals:

$$1) \int \frac{x^2}{\sqrt{1-x^6}} dx$$

$$6) \int \frac{2dx}{\sqrt{x(1+x)}}$$

$$2) \int \frac{dx}{\sqrt{9-x^2}}$$

$$7) \int \frac{dx}{1+3x^2}$$

$$3) \int \frac{x}{1+x^4} dx$$

$$8) \int \frac{2\cos x}{1+\sin^2 x} dx$$

$$4) \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx$$

$$9) \int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$5) \int \frac{dx}{x\sqrt{4x^2-1}}$$

$$10) \int \frac{\tan^{-1} x}{1+x^2} dx$$

Sol.-

$$1) \frac{1}{3} \int \frac{1}{\sqrt{1-(x^3)^2}} (3x^2 dx) = \frac{1}{3} \sin^{-1} x^3 + c$$

$$2) \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1} \frac{x}{3} + c$$

$$3) \frac{1}{2} \int \frac{2x}{1+(x^2)^2} dx = \frac{1}{2} \tan^{-1} x^2 + c$$

$$4) \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx = \sin^{-1}(\tan x) + c$$

$$5) \int \frac{2 dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$$

$$6) \int \frac{2}{\sqrt{x}(1+x)} dx = 4 \int \frac{1/2\sqrt{x} dx}{1+(\sqrt{x})^2} = 4 \tan^{-1} \sqrt{x} + c$$

$$7) \frac{1}{\sqrt{3}} \int \frac{\sqrt{3} dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}x) + c$$

$$8) 2 \int \frac{\cos x dx}{1+(\sin x)^2} = 2 \tan^{-1}(\sin x) + c$$

$$9) \int e^{\sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

$$10) \int \tan^{-1} x \cdot \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

5-4- Integrals of hyperbolic functions:

The integration formulas for the hyperbolic functions are:

$$19) \int \sinh u \cdot du = \cosh u + c$$

$$20) \int \cosh u \cdot du = \sinh u + c$$

$$21) \int \tanh u \cdot du = \ln(\cosh u) + c$$

$$22) \int \coth u \cdot du = \ln(\sinh u) + c$$

$$23) \int \sec^2 u \cdot du = \tanh u + c$$

$$24) \int \csc^2 u \cdot du = \coth u + c$$

$$25) \int \sec hu \cdot \tanh u \cdot du = -\sec hu + c$$

$$26) \int \csc hu \cdot \coth u \cdot du = -\csc hu + c$$

EX-4 – Evaluate the following integrals:

$$1) \int \frac{\cosh(\ln x)}{x} dx$$

$$2) \int \sinh(2x + 1) dx$$

$$3) \int \frac{\sinh x}{\cosh^4 x} dx$$

$$4) \int x \cdot \cosh(3x^2) dx$$

$$5) \int \sinh^4 x \cdot \cosh x dx$$

$$6) \int \operatorname{sech}^2(2x - 3) dx$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$8) \int (e^{ax} - e^{-ax}) dx$$

$$9) \int \frac{\sinh x}{1 + \cosh x} dx$$

$$10) \int \operatorname{csch}^2 x \cdot \coth x dx$$

Sol.

$$1) \int \cosh(\ln x) \cdot \left(\frac{dx}{x} \right) = \sinh(\ln x) + c$$

$$2) \frac{1}{2} \int \sinh(2x + 1) \cdot (2 dx) = \frac{1}{2} \cosh(2x + 1) + c$$

$$3) \int \frac{1}{\cosh^3 x} \cdot \frac{\sinh x}{\cosh x} dx = \int \operatorname{sech}^3 x \cdot \tanh x dx$$

$$= - \int \operatorname{sech}^2 x \cdot (-\operatorname{sech} x \cdot \tanh x dx) = -\frac{\operatorname{sech}^3 x}{3} + c$$

$$4) \frac{1}{6} \int \cosh(3x^2) \cdot (6x dx) = \frac{1}{6} \sinh(3x^2) + c$$

$$5) \int \sinh^4 x \cdot (\cosh x dx) = \frac{\sinh^5 x}{5} + c$$

$$6) \frac{1}{2} \int \operatorname{sech}^2(2x - 3) \cdot (2 dx) = \frac{1}{2} \tanh(2x - 3) + c$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \tanh x dx = \ln(\cosh x) + c$$

$$8) 2 \int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a} \int \sinh ax (a dx) = \frac{2}{a} \cosh ax + c$$

$$9) \int \frac{\sinh x dx}{1 + \cosh x} = \ln(1 + \cosh x) + c$$

$$10) - \int \operatorname{csch} x \cdot (-\operatorname{csch} x \cdot \coth x dx) = -\frac{\operatorname{csch}^2 x}{2} + c$$

5-5- Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27) \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28) \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29) \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u| < 1 \\ \coth^{-1} u + c & \text{if } |u| > 1 \end{cases} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + c$$

$$30) \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1} \left(\frac{1}{|u|} \right) + c$$

$$31) \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1} \left(\frac{1}{|u|} \right) + c$$

EX-4 – Evaluate the following integrals:

$$1) \int \frac{dx}{\sqrt{1+4x^2}}$$

$$2) \int \frac{dx}{\sqrt{4+x^2}}$$

$$3) \int \frac{dx}{1-x^2}$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}}$$

$$5) \int \frac{\sec^2 \theta \, d\theta}{\sqrt{\tan^2 \theta - 1}}$$

$$6) \int \tanh^{-1}(\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})}$$

Sol.-

$$1) \frac{1}{2} \int \frac{2 \, dx}{\sqrt{1+4x^2}} = \frac{1}{2} \sinh^{-1} 2x + c$$

$$2) \int \frac{\frac{1}{2} \, dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1} \frac{x}{2} + c$$

$$3) \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad \text{if } |x| < 1 \\ = \coth^{-1} x + c \quad \text{if } |x| > 1$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2} \int \frac{\frac{1}{2} dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2} \operatorname{cosech}^{-1} \left| \frac{x}{2} \right| + c$$

$$5) \int \frac{1}{\sqrt{\tan^2 \theta - 1}} (\sec^2 \theta d\theta) = \operatorname{cosh}^{-1} (\tan \theta) + c$$

$$6) \text{ let } u = \ln \sqrt{x} = \frac{1}{2} \ln x \quad du = \frac{1}{2x} dx$$

$$\begin{aligned} \int \tanh^{-1} (\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})} &= \int \tanh^{-1} u \cdot \frac{2 du}{1-u^2} \\ &= 2 \frac{(\tanh^{-1} u)^2}{2} + c = \left[\tanh^{-1} (\ln \sqrt{x}) \right]^2 + c \end{aligned}$$

Problems – 5

Evaluate the following integrals:

- | | |
|--|--|
| 1) $\int (x^2 - 1) \cdot (4 - x^2) dx$ | (ans. : $\frac{5}{3}x^3 - \frac{1}{5}x^5 - 4x + c$) |
| 2) $\int e^x \cdot \sin e^x dx$ | (ans. : $-\cos e^x + c$) |
| 3) $\int \tan(3x + 5) dx$ | (ans. : $-\frac{1}{3} \ln \cos(3x + 5) + c$) |
| 4) $\int \frac{\cot(\ln x)}{x} dx$ | (ans. : $\ln \sin(\ln x) + c$) |
| 5) $\int \frac{\sin x + \cos x}{\cos x} dx$ | (ans. : $-\ln \cos x + x + c$) |
| 6) $\int \frac{dx}{1 + \cos x}$ | (ans. : $-\cot x + \csc x + c$) |
| 7) $\int \cot(2x + 1) \cdot \csc^2(2x + 1) dx$ | (ans. : $-\frac{1}{4} \cot^2(2x + 1) + c$) |
| 8) $\int \frac{dx}{\sqrt{1 - 9x^2}}$ | (ans. : $\frac{1}{3} \sin^{-1}(3x) + c$) |
| 9) $\int \frac{dx}{\sqrt{2 - x^2}}$ | (ans. : $\sin^{-1} \frac{x}{\sqrt{2}} + c$) |
| 10) $\int e^{2x} \cdot \cos e^{2x} dx$ | (ans. : $\frac{1}{2} \sinh e^{2x} + c$) |
| 11) $\int e^{\sin x} \cdot \cos x dx$ | (ans. : $e^{\sin x} + c$) |
| 12) $\int \frac{dx}{e^{3x}}$ | (ans. : $-\frac{1}{3} e^{-3x} + c$) |
| 13) $\int \frac{e^{\sqrt{x}} - 1}{\sqrt{x}} dx$ | (ans. : $2e^{\sqrt{x}} - 2\sqrt{x} + c$) |
| 14) $\int x(a + b\sqrt{3x}) dx$ where a, b constants | (ans. : $\frac{1}{10} (5ax^2 + 4\sqrt{3}bx^{5/2}) + c$) |
| 15) $\int \frac{dx}{-1 - x^2}$ | (ans. : $-\tan^{-1} x + c$) |
| 16) $\int \frac{\cos \theta d\theta}{1 + \sin^2 \theta}$ | (ans. : $\tan^{-1}(\sin \theta) + c$) |

- 17) $\int \frac{1}{x^2} \csc \frac{1}{x} \cot \frac{1}{x} dx$ (ans.: $\csc \frac{1}{x} + c$)
- 18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}} dx$ (ans.: $\frac{3}{4} \sqrt[3]{(3x^2+2x+1)^2} + c$)
- 19) $\int \sin(\tan \theta) \cdot \sec^2 \theta d\theta$ (ans.: $-\cos(\tan \theta) + c$)
- 20) $\int \sqrt{x^2 - x^4} dx$ (ans.: $-\frac{1}{3} \sqrt{(1-x^2)^3} + c$)
- 21) $\int \frac{\sec^2 2x dx}{\sqrt{\tan 2x}}$ (ans.: $\sqrt{\tan 2x} + c$)
- 22) $\int (\sin \theta - \cos \theta)^2 d\theta$ (ans.: $\theta + \cos^2 \theta + c$)
- 23) $\int \frac{y}{y^4 + 1} dy$ (ans.: $\frac{1}{2} \tan^{-1} y^2 + c$)
- 24) $\int \frac{dx}{\sqrt{x(x+1)}}$ (ans.: $2 \tan^{-1} \sqrt{x} + c$)
- 25) $\int t^{\frac{2}{3}} (t^{\frac{5}{3}} + 1)^{\frac{2}{3}} dt$ (ans.: $\frac{9}{25} (t^{\frac{5}{3}} + 1)^{\frac{5}{3}} + c$)
- 26) $\int \frac{dx}{x^{\frac{1}{5}} \sqrt{1+x^{\frac{4}{5}}}}$ (ans.: $\frac{5}{2} \sqrt{1+x^{\frac{4}{5}}} + c$)
- 27) $\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}} dx$ (ans.: $-\frac{1}{12} (\cos^{-1} 4x)^3 + c$)
- 28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ (ans.: $\sec^{-1}(2x) + c$)
- 29) $\int \frac{dx}{(e^x + e^{-x})^2}$ (ans.: $\frac{1}{4} \tanh x + c$)
- 30) $\int 3^{\ln x^2} \frac{dx}{x}$ (ans.: $\frac{1}{2 \ln 3} 3^{\ln x^2} + c$)
- 31) $\int \frac{\cot x dx}{\ln(\sin x)}$ (ans.: $\ln \ln(\sin x) + c$)
- 32) $\int \frac{(\ln x)^2}{x} dx$ (ans.: $\frac{1}{3} (\ln x)^3 + c$)
- 33) $\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x} dx$ (ans.: $e^{\sec x} + c$)

- 34) $\int \frac{dx}{x \cdot \ln x}$ (ans. : $\ln \ln x + c$)
- 35) $\int \frac{d\theta}{\cosh \theta + \sinh \theta}$ (ans. : $-e^{-\theta} + c$)
- 36) $\int \frac{2^x - 8^{2x}}{\sqrt{4^x}} dx$ (ans. : $x - \frac{1}{5 \ln 2} 2^{5x} + c$)
- 37) $\int \frac{e^{\tan^{-1} 2t}}{1 + 4t^2} dt$ (ans. : $\frac{1}{2} e^{\tan^{-1} 2t} + c$)
- 38) $\int \frac{\cot x}{\csc x} dx$ (ans. : $\sin x + c$)
- 39) $\int \sec^4 x \cdot \tan^3 x dx$ (ans. : $\frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + c$)
- 40) $\int \csc^4 3x dx$ (ans. : $-\frac{1}{9} \cot^3 3x - \frac{1}{3} \cot 3x + c$)
- 41) $\int \frac{\cos^3 t}{\sin^2 t} dt$ (ans. : $-\operatorname{csc} t - \sin t + c$)
- 42) $\int \frac{\sec^4 x}{\tan^4 x} dx$ (ans. : $-\frac{1}{3} \cot^3 x - \cot x + c$)
- 43) $\int \tan^2 4\theta d\theta$ (ans. : $\frac{1}{4} \tan 4\theta - \theta + c$)
- 44) $\int \frac{e^x}{1 + e^x} dx$ (ans. : $\ln(1 + e^x) + c$)
- 45) $\int \tan^3 2x dx$ (ans. : $\frac{1}{4} \tan^2 2x + \frac{1}{2} \ln|\cos 2x| + c$)
- 46) $\int \frac{\sec^2 x}{2 + \tan x} dx$ (ans. : $\ln(2 + \tan x) + c$)
- 47) $\int \sec^4 3x dx$ (ans. : $\frac{1}{9} \tan^3 3x + \frac{1}{3} \tan 3x + c$)
- 48) $\int \frac{e^t}{1 + e^{2t}} dt$ (ans. : $\tan^{-1} e^t + c$)
- 49) $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ (ans. : $2 \sin \sqrt{x} + c$)
- 50) $\int \frac{dx}{\sin x \cdot \cos x}$ (ans. : $-\ln|\csc 2x + \cot 2x| + c$)

$$51) \int \sqrt{1 + \sin y} \, dy \quad (\text{ans. : } -2\sqrt{1 - \sin y} + c)$$

$$52) \int \frac{dx}{(x^2 + 1)(2 + \tan^{-1} x)} \quad (\text{ans. : } \ln(2 + \tan^{-1} x) + c)$$

$$53) \int \sin^{-1}(\cosh x) \cdot \frac{\sinh x \, dx}{\sqrt{1 - \cosh^2 x}} \quad (\text{ans. : } \frac{1}{2} (\sinh^{-1}(\cosh x))^2 + c)$$

$$54) \int \frac{\cos \theta \, d\theta}{1 - \sin^2 \theta} \quad (\text{ans. : } \ln|\sec \theta + \tan \theta| + c)$$

$$55) \int \frac{dx}{x(1 + (\ln x)^2)} \quad (\text{ans. : } \tan^{-1}(\ln x) + c)$$

$$56) \int \left(e^{\frac{2}{4}x} - 2e^{\frac{5}{4}x} + e^{\frac{x}{4}} \right) dx \quad (\text{ans. : } \frac{4}{9}e^{\frac{2}{4}x} - \frac{8}{5}e^{\frac{5}{4}x} + 4e^{\frac{x}{4}} + c)$$

$$57) \int \frac{e^x \, dx}{e^{2x} + 2e^x + 1} \quad (\text{ans. : } -\frac{1}{e^x + 1} + c)$$

$$58) \int e^x \cdot \sinh 2x \, dx \quad (\text{ans. : } \frac{1}{2} \left[\frac{1}{3}e^{3x} + e^{-x} \right] + c)$$

$$59) \int \frac{\sec^3 x + e^{\sin x}}{\sec x} \, dx \quad (\text{ans. : } \tan x + e^{\sin x} + c)$$

$$60) \int \frac{3^{x+2}}{2 + 9^{x+1}} \, dx \quad (\text{ans. : } \frac{3}{\sqrt{2} \ln 3} \tan^{-1} \frac{3^{x+1}}{\sqrt{2}} + c)$$

$$61) \int \frac{\cos x \, dx}{\sqrt{\sin x} \cdot \sqrt{1 - \sin x}} \quad (\text{ans. : } 2\sin^{-1} \sqrt{\sin x} + c)$$

$$62) \int \tan^5 x \, dx \quad (\text{ans. : } \frac{1}{4} \sec^4 x - \sec^2 x - \ln|\cos x| + c)$$

$$63) \int e^{\ln \sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} \quad (\text{ans. : } \frac{1}{2} (\sin^{-1} x)^2 + c)$$

$$64) \int x \cdot e^{x^2-1} \, dx \quad (\text{ans. : } \frac{1}{2} e^{x^2-1} + c)$$

$$65) \int \cosh(\ln \cos x) \, dx \quad (\text{ans. : } \frac{1}{2} [\sin x + \ln|\sec x + \tan x|] + c)$$

$$66) \int \frac{\cos x}{\sin^2 x} \, dx \quad (\text{ans. : } -\csc x + c)$$

$$67) \int \cosh^{-1}(\sin x) \frac{\cos x \, dx}{\sqrt{\sin^2 x - 1}} \quad (\text{ans. : } \frac{1}{2} [\cosh^{-1}(\sin x)]^2 + c)$$