جامعة المستقبل
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Course: Digital Signal Processing

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Lecture 17 & 18

(Z - Transform)

Z - Transform

17th&18th modular units

1/ Overview

1 / A –Target population :-

For students of third class

Department of Medical Instrumentation Eng. Techniques

1 / B –Rationale :-

The z-transform is a useful tool in the analysis of discrete-time signals and systems and is the discrete-time counterpart of the Laplace transform for continuous-time signals and systems. The z-transform may be used to solve constant coefficient difference equations, evaluate the response of a linear time-invariant system to a given input, and design linear filters.

1 / C –Central Idea :-

This unit introduces the z-transform and its properties; illustrates how to determine the inverse z-transform using partial fraction expansion; and applies the z-transform to solve linear difference equations.

2/ Performance Objectives :-

After studying the 9th modular unit, the student will be able to:-

1. Define z transform.
2. Know the methods of determining z transform.
3. Know the methods of determining inverse z transform.

- Solving difference equations using 2 and 3.

3/ the text :-

The z-Transform

Definition

The z-transform is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals. We begin with the definition of the z-transform.

The z-transform of a causal sequence *x(n)*, designated by *X(z)* or *Z(x(n))*, is defined as

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n)z^{-n}$$
$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

where z is the complex variable. Here, the summation taken from $n = 0$ to $n = \infty$ is according to the fact that for most situations, the digital signal $x(n)$ is the causal sequence, that is, $x(n) = 0$ for $n < 0$.

Thus, the definition above is referred to as a one-sided z-transform or a unilateral transform.

Example

Given the sequence

$$x(n) = u(n),$$

Find the z-transform of *x(n)*.

Solution:

From the definition, the z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = 1 + (z^{-1}) + (z^{-1})^2 + \ldots .$$

This is an infinite geometric series that converges to

$$X(z) = \frac{z}{z-1}$$

The region of convergence for all values of z is given as $|z| > 1$

Example

Considering the exponential sequence

$$x(n) = a^n u(n),$$

Find the z-transform of the sequence *x(n)*.

Solution:

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = 1 + (az^{-1}) + (az^{-1})^2 + \ldots .$$

Since this is a geometric series which will converge for $|az^{-1}| < 1$ it is further expressed as

$$X(z) = \frac{z}{z-a}, \text{ for } |z| > |a|.$$

TABLE 5.1 Table of z-transform pairs.

Line No.	$x(n), n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n) z^{-n}$	
2	$\delta(n)$	1	$\|z\| > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\|z\| > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
5	$n^2 u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
7	$e^{-na} u(n)$	$\frac{z}{(z-e^{-a})}$	$\|z\| > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
11	$a^n \sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\|z\| > \|a\|$
12	$a^n \cos(bn)u(n)$	$\frac{z[z - a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\|z\| > \|a\|$
13	$e^{-an} \sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
14	$e^{-an} \cos(bn)u(n)$	$\frac{z[z - e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
15	$2\|A\|\|P\|^n \cos(n\theta + \phi)u(n)$ where P and A are complex constants defined by $P = \|P\|\angle\theta, A = \|A\|\angle\phi$	$\frac{Az}{z-P} + \frac{A^* z}{z-P^*}$	

Example

Find the z-transform for each of the following sequences:

a. $x(n) = 10u(n)$
b. $x(n) = 10 \sin(0.25\pi n)u(n)$
c. $x(n) = (0.5)^n u(n)$
d. $x(n) = (0.5)^n \sin(0.25\pi n)u(n)$
e. $x(n) = e^{-0.1n} \cos(0.25\pi n)u(n)$

Solution:

a. From Line 3 in Table 5.1, we get

$$X(z) = Z(10u(n)) = \frac{10z}{z-1}.$$

b. Line 9 in Table 5.1 leads to

$$X(z) = 10Z(\sin(0.2\pi n)u(n))$$
$$= \frac{10\sin(0.25\pi)z}{z^2 - 2z\cos(0.25\pi) + 1} = \frac{7.07z}{z^2 - 1.414z + 1}.$$

c. From Line 6 in Table 5.1, we yield

$$X(z) = Z((0.5)^n u(n)) = \frac{z}{z-0.5}.$$

d. From Line 11 in Table 5.1, it follows that

$$X(z) = Z((0.5)^n \sin(0.25\pi n)u(n)) = \frac{0.5 \times \sin(0.25\pi)z}{z^2 - 2 \times 0.5\cos(0.25\pi)z + 0.5^2}$$
$$= \frac{0.3536z}{z^2 - 1.4142z + 0.25}.$$

e. From Line 14 in Table 5.1, it follows that

$$X(z) = Z(e^{-0.1n}\cos(0.25\pi n)u(n)) = \frac{z(z - e^{-0.1}\cos(0.25\pi))}{z^2 - 2e^{-0.1}\cos(0.25\pi)z + e^{-0.2}}$$
$$= \frac{z(z-0.6397)}{z^2 - 1.2794z + 0.8187}.$$

Properties of the z-Transform

Linearity: The z-transform is a linear transformation, which implies

$$Z(ax_1(n) + bx_2(n)) = aZ(x_1(n)) + bZ(x_2(n)),$$

where $x_1(n)$ and $x_2(n)$ denote the sampled sequences, while a and b are the arbitrary constants.

Example

Find the z-transform of the sequence defined by

$$x(n) = u(n) - (0.5)^n u(n).$$

Solution:

Applying the linearity of the z-transform previously discussed, we have

$$x(n) = u(n) - (0.5)^n u(n).$$

Using Table 5.1 yields

$$Z(u(n)) = \frac{z}{z-1}$$

$$\text{and } Z(0.5^n u(n)) = \frac{z}{z-0.5}.$$

Substituting these results into X(z) leads to the final solution,

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}.$$

Shift theorem: Given $X(z)$, the z-transform of a sequence $x(n)$, the z-transform of $x(n - m)$, the time-shifted sequence, is given by

$$Z(x(n-m)) = z^{-m}X(z).$$

Example

Determine the z-transform of the following sequence:

$$y(n) = (0.5)^{(n-5)} \cdot u(n-5),$$

where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n < 5$.

Solution:

We first use the shift theorem to have

$$Y(z) = Z\left[(0.5)^{n-5}u(n-5)\right] = z^{-5}Z[(0.5)^n u(n)].$$

Using Table 5.1 leads to

$$Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}.$$

Convolution: Given two sequences x1(n) and x2(n), their convolution can be determined as follows:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k) x_2(k),$$

In z-transform domain, we have

$$X(z) = X_1(z) X_2(z).$$

Here, $X(z) = Z(x(n))$, $X_1(z) = Z(x_1(n))$, and $X_2(z) = Z(x_2(n))$.

Example

Given two sequences,

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$
$$x_2(n) = 2\delta(n) - \delta(n-1),$$

a) Find the z-transform of their convolution:

$$X(z) = Z(x_1(n) * x_2(n)).$$

b) Determine the convolution sum using the z-transform:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(k) x_2(n-k).$$

Solution:

a) Applying z-transform to $x_1(n)$ and $x_2(n)$, respectively, it follows that

$$X_1(z) = 3 + 2z^{-1}$$
$$X_2(z) = 2 - z^{-1}.$$

Using the convolution property, we have

$$X(z) = X_1(z) X_2(z) = (3 + 2z^{-1})(2 - z^{-1})$$
$$= 6 + z^{-1} - 2z^{-2}.$$

b) Applying the inverse z-transform and using the shift theorem and line 1 of Table 5.1 leads to

$$x(n) = Z^{-1}(6 + z^{-1} - 2z^{-2}) = 6\delta(n) + \delta(n-1) - 2\delta(n-2).$$

Inverse z-Transform

The z-transform of the sequence x(n) and the inverse z-transform of the function X(z) are defined as, respectively,

$$X(z) = Z(x(n))$$
$$\text{and } x(n) = Z^{-1}(X(z)),$$

Where Z( ) is the z-transform operator, while Z^{-1}( ) is the inverse z-transform operator.

The inverse z-transform may be obtained by at least three methods:

1. Partial fraction expansion and look-up table.
2. Power series expansion.
3. Residue method.

Example

Find the inverse z-transform for each of the following functions:

a. $X(z) = 2 + \frac{4z}{z-1} - \frac{z}{z-0.5}$

b. $X(z) = \frac{5z}{(z-1)^2} - \frac{2z}{(z-0.5)^2}$

c. $X(z) = \frac{10z}{z^2 - z + 1}$

d. $X(z) = \frac{z^{-4}}{z-1} + z^{-6} + \frac{z^{-3}}{z+0.5}$

Solution:

a. $x(n) = 2Z^{-1}(1) + 4Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{z-0.5}\right)$.

From Table 5.1, we have

$x(n) = 2\delta(n) + 4u(n) - (0.5)^n u(n)$.

b. $x(n) = Z^{-1}\left(\frac{5z}{(z-1)^2}\right) - Z^{-1}\left(\frac{2z}{(z-0.5)^2}\right) = 5Z^{-1}\left(\frac{z}{(z-1)^2}\right) - \frac{2}{0.5}Z^{-1}\left(\frac{0.5z}{(z-0.5)^2}\right)$.

Then $x(n) = 5nu(n) - 4n(0.5)^n u(n)$.

c. Since $X(z) = \frac{10z}{z^2 - z + 1} = \left(\frac{10}{\sin(a)}\right)\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}$,

by coefficient matching, we have

$$-2\cos(a) = -1.$$

Hence, $\cos(a) = 0.5$, and $a = 60°$. Substituting $a = 60°$ into the sine function leads to

$$\sin(a) = \sin(60°) = 0.866.$$

Finally, we have

$$x(n) = \frac{10}{\sin(a)} Z^{-1}\left(\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \frac{10}{0.866}\sin(n \cdot 60^0)$$

$$= 11.547\sin(n \cdot 60^0).$$

d. Since

$$x(n) = Z^{-1}\left(z^{-5}\frac{z}{z-1}\right) + Z^{-1}(z^{-6} \cdot 1) + Z^{-1}\left(z^{-4}\frac{z}{z+0.5}\right),$$

using Table 5.1 and the shift property, we get

$$x(n) = u(n-5) + \delta(n-6) + (-0.5)^{n-4}u(n-4).$$

Now, we are ready to deal with the inverse z-transform using the partial fraction expansion and look-up table. The general procedure is as follows:

1. Eliminate the negative powers of z for the z-transform function X(z).
2. Determine the rational function X(z)/z (assuming it is proper), and apply the partial fraction expansion to the determined rational function X(z)/z using the formula in the following Table.

TABLE 5.3 Partial fraction(s) and formulas for constant(s).

Partial fraction with the first-order real pole:

$$\frac{R}{z-p} \qquad R = (z-p)\frac{X(z)}{z}\bigg|_{z=p}$$

Partial fraction with the first-order complex poles:

$$\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)} \qquad A = (z-P)\frac{X(z)}{z}\bigg|_{z=P}$$

P^* = complex conjugate of P

A^* = complex conjugate of A

Partial fraction with mth-order real poles:

$$\frac{R_m}{(z-p)} + \frac{R_{m-1}}{(z-p)^2} + \cdots + \frac{R_1}{(z-p)^m} \qquad R_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \frac{X(z)}{z}\right)\bigg|_{z=p}$$

3. Multiply the expanded function X(z)/z by z on both sides of the equation to obtain X(z).
4. Apply the inverse z-transform using Table 5.1.

Example

Find the inverse of the following z-transform:

$$X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}.$$

Solution:

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1-z^{-1})(1-0.5z^{-1})}.$$

$$= \frac{z^2}{(z-1)(z-0.5)}$$

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}.$$

Then A and B are constants found using the formula in Table 5.3, that is,

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

Using Table 5.1 of the z-transform pairs, it follows that

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Example

Find $y(n)$ if

$$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)}.$$

Solution:

Dividing $Y(z)$ by z, we have

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2-z+0.5)}.$$

Applying the partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}.$$

We first find B:

$$B = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\bigg|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4.$$

Notice that A and A*form a complex conjugate pair. We determine A asfollows:

$$A = (z - 0.5 - j0.5)\frac{Y(z)}{z}\bigg|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\bigg|_{z=0.5+j0.5}$$

$$A = \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}.$$

Using the polar form, we get

$$A = \frac{(0.707\angle 45^\circ)(1.58114\angle 18.43^\circ)}{(0.707\angle 135^\circ)(1\angle 90^\circ)} = 1.58114\angle -161.57^\circ$$

$$A^* = \bar{A} = 1.58114\angle 161.57^\circ.$$

Assume that a first-order complex pole has the form

$$P = 0.5 + 0.5j = |P|\angle\theta = 0.707\angle 45^\circ \text{ and } P^* = |P|\angle -\theta = 0.707\angle -45^\circ.$$

We have

$$Y(z) = \frac{4z}{z-1} + \frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}.$$

Applying the inverse z-transform from line 15 in Table 5.1 leads to

$$y(n) = 4Z^{-1}\left(\frac{z}{z-1}\right) + Z^{-1}\left(\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}\right).$$

Using the previous formula, the inversion and subsequent simplification yield

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n \cos(n\theta + \phi)u(n) \\ &= 4u(n) + 3.1623(0.7071)^n \cos(45^\circ n - 161.57^\circ)u(n) \end{aligned}$$

Example

Find *x(n)* if

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2}.$$

Solution:

Dividing both sides of the previous z-transform by z yields

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)^2} = \frac{A}{z-1} + \frac{B}{z-0.5} + \frac{C}{(z-0.5)^2},$$

$$\text{where } A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)^2}\bigg|_{z=1} = 4.$$

$$B = R_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$
$$= \frac{d}{dz}\left(\frac{z}{z-1}\right)\Big|_{z=0.5} = \frac{-1}{(z-1)^2}\Big|_{z=0.5} = -4$$

$$C = R_1 = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$
$$= \frac{z}{z-1}\Big|_{z=0.5} = -1.$$

Then $X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2}.$

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n),$$
$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n),$$
$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n).$$

From these results, it follows that

$$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n).$$

Solution of Difference Equations Using the z-Transform

To solve a difference equation with initial conditions, we have to deal with time shifted sequences such as y(n - 1), y(n - 2), . . . , y(n - m), and so on. Let us examine the z-transform of these terms. Using the definition of the z-transform, we have

$$Z(y(n-1)) = \sum_{n=0}^{\infty} y(n-1)z^{-n}$$
$$= y(-1) + y(0)z^{-1} + y(1)z^{-2} + \ldots$$
$$= y(-1) + z^{-1}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \ldots\right)$$

It holds that

$$Z(y(n-1)) = y(-1) + z^{-1}Y(z).$$

Similarly, we can have

$$
\begin{aligned}
Z(y(n-2)) &= \sum_{n=0}^{\infty} y(n-2)z^{-n} \\
&= y(-2) + y(-1)z^{-1} + y(0)z^{-2} + y(1)z^{-3} + \ldots \\
&= y(-2) + y(-1)z^{-1} + z^{-2}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \ldots\right) \\
Z(y(n-2)) &= y(-2) + y(-1)z^{-1} + z^{-2}Y(z) \\
Z(y(n-m)) &= y(-m) + y(-m+1)z^{-1} + \ldots + y(-1)z^{-(m-1)} \\
&\quad + z^{-m}Y(z),
\end{aligned}
$$

where y(- m), y(- m + 1), . . . , y(- 1) are the initial conditions. If all initial conditions are considered to be zero, that is,

$$y(-m) = y(-m+1) = \ldots y(-1) = 0,$$

Then

$$Z(y(n-m)) = z^{-m}Y(z),$$

The following two examples serve as illustrations of applying the z-transform to find the solutions of the difference equations. The procedure is:

1. Apply z-transform to the difference equation.
2. Substitute the initial conditions.
3. Solve for the difference equation in z-transform domain.
4. Find the solution in time domain by applying the inverse z-transform.

Example

A digital signal processing (DSP) system is described by the difference equation

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n).$$

Determine the solution when the initial condition is given by y(-1) = 1.

Solution:

Applying the z-transform on both sides of the difference equation

$$Y(z) - 0.5\left(y(-1) + z^{-1}Y(z)\right) = 5Z(0.2^n u(n)).$$

Substituting the initial condition and $Z(0.2^n u(n)) = z/(z-0.2)$, we achieve

$$Y(z) - 0.5\left(1 + z^{-1}Y(z)\right) = 5z/(z-0.2).$$

Simplification yields

$$Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z-0.2).$$

Factoring out Y(z) and combining the right-hand side of the equation, it follows that

$$Y(z) = \frac{(5.5z - 0.1)}{(1 - 0.5z^{-1})(z - 0.2)} = \frac{z(5.5z - 0.1)}{(z - 0.5)(z - 0.2)}.$$

Then we obtain

$$Y(z) = \frac{(5.5z - 0.1)}{(1 - 0.5z^{-1})(z - 0.2)} = \frac{z(5.5z - 0.1)}{(z - 0.5)(z - 0.2)}.$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{5.5z - 0.1}{(z - 0.5)(z - 0.2)} = \frac{A}{z - 0.5} + \frac{B}{z - 0.2},$$

$$A = (z - 0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z - 0.1}{z - 0.2}\bigg|_{z=0.5} = \frac{5.5 \times 0.5 - 0.1}{0.5 - 0.2} = 8.8333,$$

$$B = (z - 0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z - 0.1}{z - 0.5}\bigg|_{z=0.2} = \frac{5.5 \times 0.2 - 0.1}{0.2 - 0.5} = -3.3333.$$

Thus

$$Y(z) = \frac{8.8333z}{(z - 0.5)} + \frac{-3.3333z}{(z - 0.2)},$$

which gives the solution as

$$y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n).$$

4/Sources :-

1. Schaum's Outline of Theory and Problems of Digital Signal processing.
2. Digital signal processing, principles, algorithms, and applications by John G. Proakis and Dimitris G. Manolakis.
3. Signal and systems, Alan Oppenheim.