جامعة المستقبل

كلية الهندسة والتقنيات الهندسية

قسم هندسة تقنيات الاجهزة الطبية

Course: Digital Signal Processing

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&

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Lecture 13 & 14

Properties of DFT

Properties of DFT

13th&14th modular units

1/ Overview

1 / A –Target population :-

For students of third class

Department of Medical Instrumentation Eng. Techniques

1 / B –Rationale :-

The DFT exhibits a number of useful properties and operational relationships that are similar to the properties of the continuous Fourier transform

1 / C –Central Idea :-

- **The DFT as a Linear Transformation**
- **Periodicity, linearity, and symmetry properties**
- **Circular Symmetries of a Sequence.**
- **Multiplication of Two DFTs and Circular Convolution**

2/ Performance Objectives :-

After studying the 7^{th} modular unit, the student will be able to:-

1. Compute DFT in different manner.
2. Know the properties of DFT.
3. Know the circular convolution.

4/ the text :-

The DFT as a Linear Transformation

The formulas for the DFT and IDFT given above may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \dots \dots \dots . . N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \dots \dots \dots . . N-1$$

Where, by definition,

$$W_N = e^{-j2\pi/N}$$

We note that the computation of each point of the DFT can be accomplished by N complex multiplications and (N-1) complex additions. Hence the N – point DFT values can be computed in a total of N^2 complex multiplications and N (N-1) complex additions.

Let us define an N – point vector $\mathbf{x}_N$ of the signal sequence *x(n), n = 0, 1,…N-1*, an N – point vector $\mathbf{X}_N$ of frequency samples, and an N × N matrix $\mathbf{W}_N$ as

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N$$

Where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$

Example

Compute the DFT of the four – point sequence

$$x(n) = [0 \quad 1 \quad 2 \quad 3]$$

Solution

The first step is to determine the matrix $\mathbf{W_4}$. By exploiting the periodicity property of $\mathbf{W_4}$ and the symmetry property

$$\mathbf{W_N^{k+N/2}} = -\mathbf{W_N^k}$$

the matrix $\mathbf{W}_4$ may be expressed as

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Properties of the DFT

In this section we present the important properties of the DFT. The notation used below to denote the *N*-point DFT pair *x(n)* and *X(k)* is

$$x(n) \overset{DFT}{\longleftrightarrow} X(k)$$

Periodicity, linearity, and symmetry properties

Periodicity. If $x(n)$ and X(k) are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k$$

Linearity. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1x_1(n) + a_2x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k)$$

Circular Symmetries of a Sequence.

As we have seen, the *N*-point DFT of a finite duration sequence, $x(n)$ of length $L \leq N$ is equivalent to the *N*- point DFT of a periodic sequence $x_p(n)$, of period *N*, which is obtained by periodically extending $x(n)$, that is

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

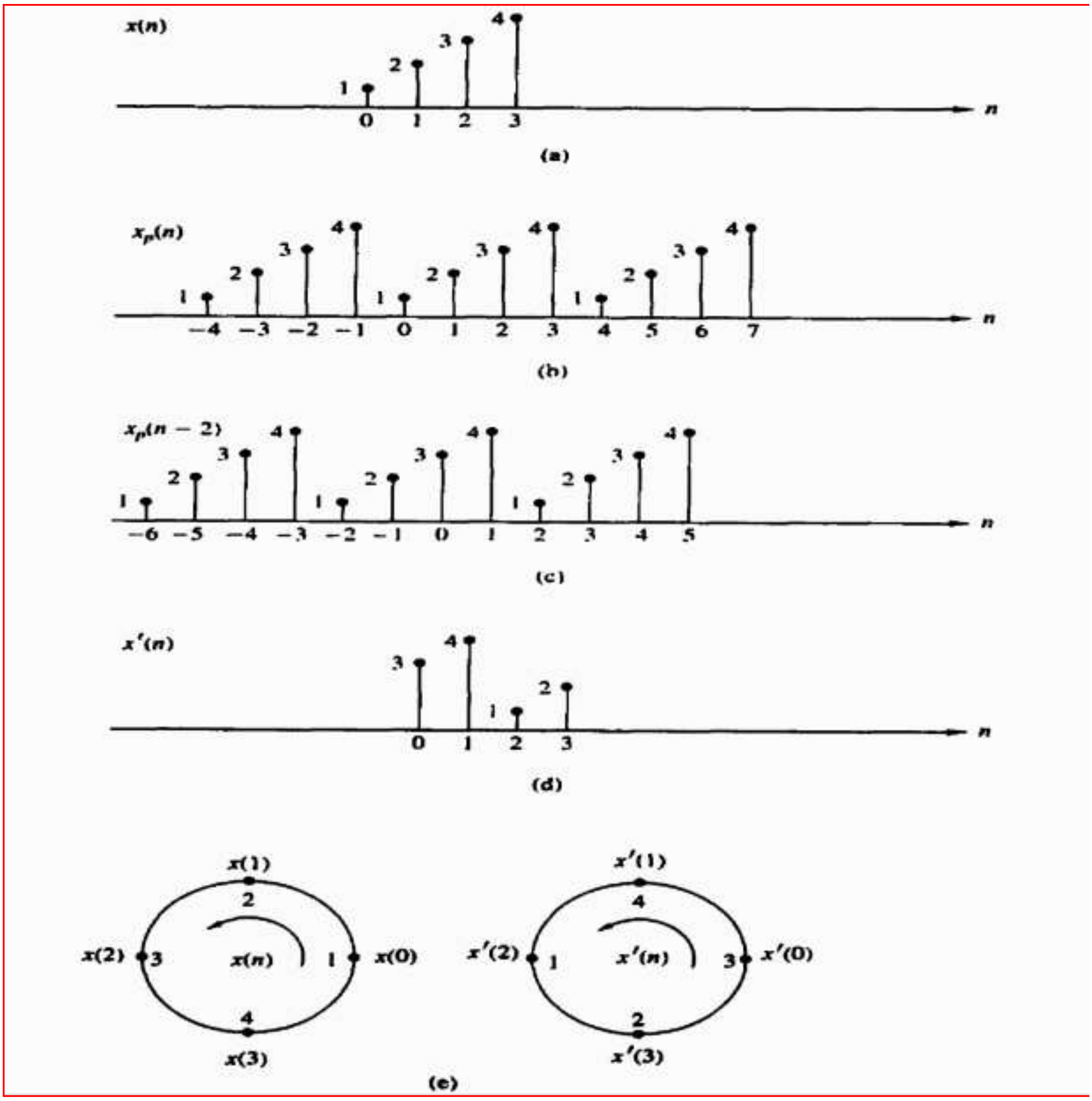
Now suppose that we shift the periodic sequence $x_p(n)$ by k units to the right. Thus we obtain another periodic sequence

$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n - k - lN)$$

The infinite – duration sequence

$$x'(n) = \begin{cases} x'_p & 0 \leq n \leq N-1 \\ 0 & otherwise \end{cases}$$

Is related to the original sequence $x(n)$ by a circular shift. This relationship is illustrated as shown in the figure below for $N=4$.



In general, the circular shift of the sequence can be represented as the index modulo N. Thus we can write

$$x'(n) = x(n-k, modulo\ N)$$
$$= x((n-k))_N$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((n-2))_4$$

Which implies that

$$x'(0) = x((-2))_4 = x(2)$$
$$x'(1) = x((-1))_4 = x(3)$$
$$x'(2) = x((0))_4 = x(0)$$
$$x'(3) = x((1))_4 = x(1)$$

Thus we conclude that a circular shift of an *N*-point sequence is equivalent to a linear shift of its periodic extension, and vice versa.

Multiplication of Two DFTs and Circular Convolution

Suppose that we have two finite – duration sequences of length *N*, $x_1(n)$ and $x_2(n)$. Their respective *N*-point DFTs are

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} \qquad k = 0, 1 \dots\dots N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N} \qquad k = 0,1 \dots\dots N-1$$

If we multiply the two DFTs together, the result is a DFT, say $X_3(k)$, of a sequence $x_3(n)$ of length *N*. Let us determine the relationship between $x_3(n)$ and the sequence $x_1(n)$ and $x_2(n)$.

We have

$$X_3(k) = X_1(k)X_2(k) \qquad k = 0, 1, \dots, N-1$$

The IDFT of $(X_3(k))$ *is*

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)e^{j2\pi km/N}$$

The inner sum in the brackets has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a = 1 \\ \dfrac{1-a^N}{1-a} & a \neq 1 \end{cases}$$

Where a is defined as

$$a = e^{j2\pi(m-n-l)/N}$$

We observe that $a = 1$ when m-n-l is a multiple of *N*. On the other hand, $a^N = 1$ for any value of $a \neq 0$. Consequently

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & l = m - n + pN = ((m-n))_N, \quad p \text{ an integer} \\ 0 & \text{otherwise} \end{cases}$$

Then the desired expression for $x_3(m)$ in the form

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N \qquad\qquad m = 0,1 \ldots\ldots N-1$$

The expression above has the form of a convolution sum. However, it is not the ordinary linear convolution. Instead, the convolution sum involves the index $((m-n))_N$ and is called circular convolution. Thus we conclude that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

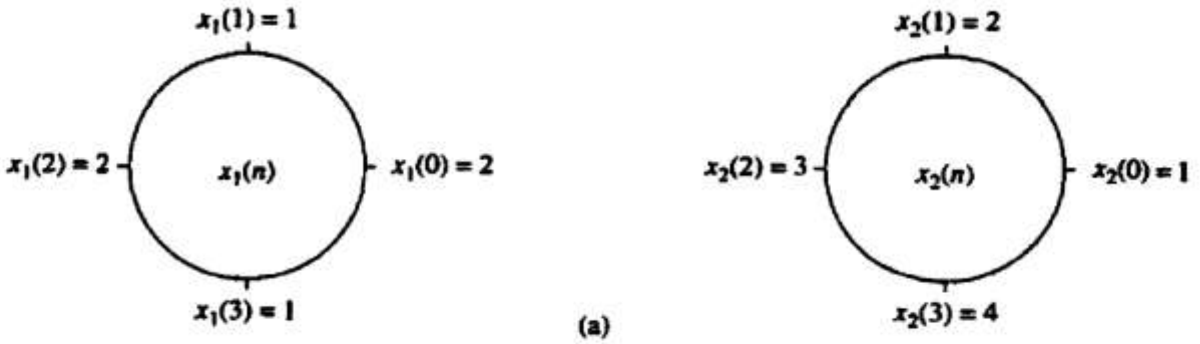
Example

Perform the circular convolution of the following two sequences:

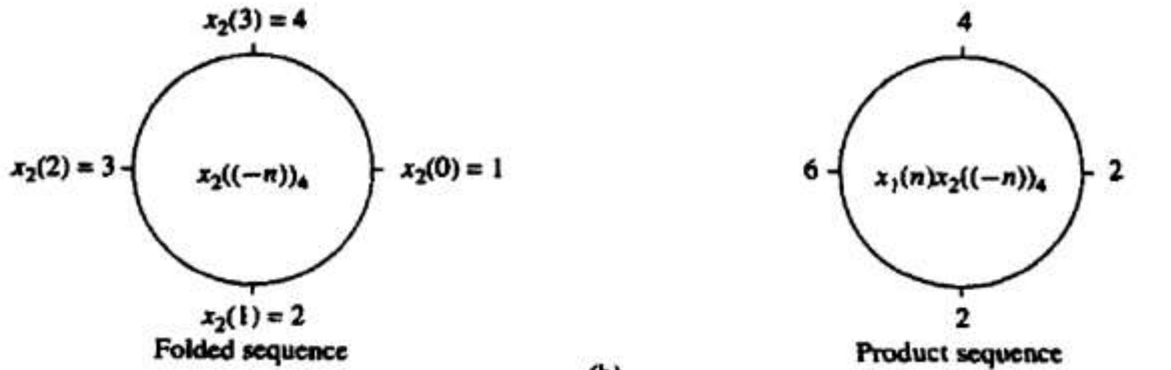
$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Solution



(a)



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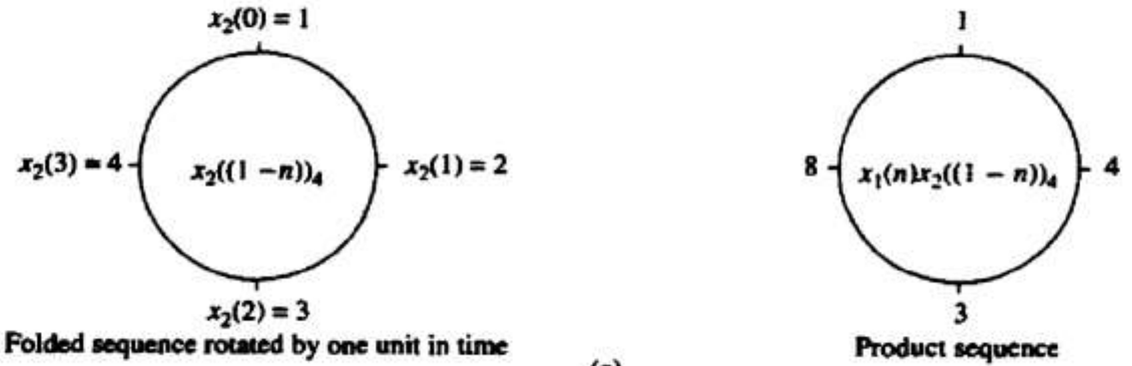

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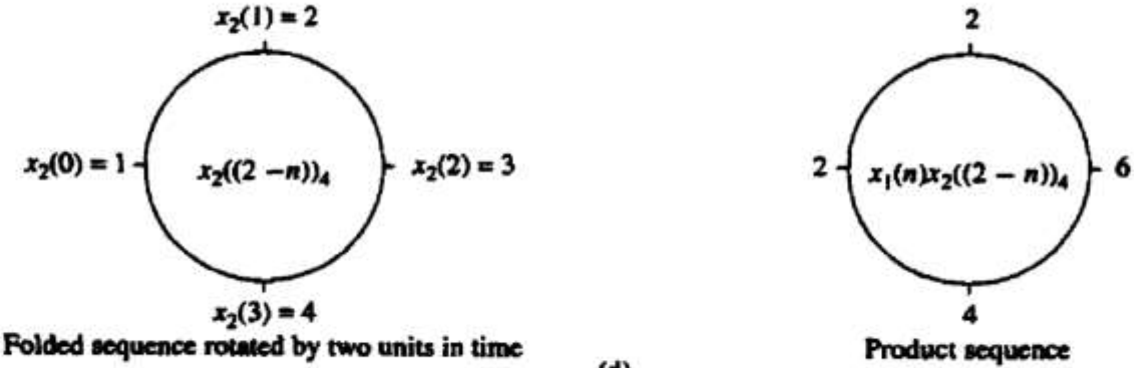

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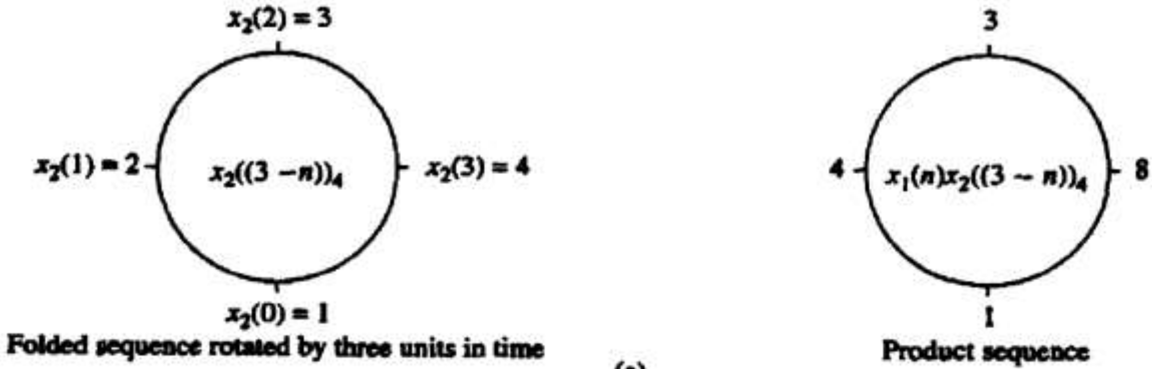

(e)

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

Example

By means of the DFT and IDFT, determine the sequence $x_3(n)$ corresponding to the circular convolution of the sequences $x_1(n)$ and $x_2(n)$ given in previous example.

Solution

First we compute the DFTs of $x_1(n)$ and $x_2(n)$. The four – point DFT of $x_1(n)$ is

$$X_1(k) = \sum_{n=0}^{3} x_1(n)e^{-j2\pi nk/4} \qquad k = 0, 1, 2, 3$$
$$= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} + e^{-j3\pi k/2}$$

Thus

$X_1(0) = 6 \qquad X_1(1) = 0 \qquad X_1(2) = 2 \qquad X_1(3) = 0$

The DFT of $x_2(n)$ is

$$X_2(k) = \sum_{n=0}^{3} x_2(n)e^{-j2\pi nk/4} \qquad k = 0, 1, 2, 3$$
$$= 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + 4e^{-j3\pi k/2}$$

Thus

$X_2(0) = 10 \qquad X_2(1) = -2 + j2 \qquad X_2(2) = -2 \qquad X_2(3)$
$= -2 - j2$

When we multiply the two DFTs, we obtain the product

$$X_3(k) = X_1(k)X_2(k)$$

Or, equivalently

$X_3(0) = 60 \qquad X_3(1) = 0 \qquad X_3(2) = -4 \qquad X_3(3) = 0$

Now, the IDFT of $X_3(k)$ is

$$x_3(n) = \sum_{k=0}^{3} X_3(k)e^{j2\pi nk/4} \qquad n = 0,1,2,3$$
$$= \frac{1}{4}\left(60 - 4e^{j\pi n}\right)$$

Thus

$x_3(0) = 14 \qquad x_3(1) = 16 \qquad x_3(2) = 14 \qquad x_3(3) = 16$

Which is the result obtained in previous example.

We conclude this section by formally stating this important property of the DFT.

Circular convolution. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n) \, \textcircled{N} \, x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)X_2(k)$$

7/Sources:-

1. Schaum's Outline of Theory and Problems of Digital Signal processing.
2. Digital signal processing, principles, algorithms, and applications by John G. Proakis and Dimitris G. Manolakis.
3. Signal and systems, Alan Oppenheim.