Al-Mustaqbal university

Engineering technical college

Department of Building &Construction Engineering

Mathematics

First class

Lecture No.10

Assist. Lecture

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The integration

Definition: A function F is called an ant derivative of a function f if the derivative of F is f.

Remark: F (x) + C , C is constant is also an antiderivatives of f (x) , where

$$\frac{d}{dx}(F(x)+c=f(x)$$

The indefinite integral The process of finding antiderivatives is called anti differently on integration. We denote this by

$$\int f(x)dx=F(x)+c$$

Since

- $\frac{d}{dx}(F(x)=f^{/}(x)$. then$\int f^{/}(x)dx=f(x)+c$
- $\frac{d}{dx}(\int f(x)dx=F(x)$

Theorem:

If f (x) and g (x) are integrals functions for each x, and m is constant, then

- $\int mf(x)dx=m\int f(x)$
- $\int[f(x)\pm g(x)]dx=\int f(x)\pm\int g(x)$
- $\int\left(\frac{f(x)^{/}}{g(x)}\right)dx=ln|f(x)|+c$

Theorem:

1. $\int dx=x+c$
2. $\int sinx\ dx=-cosx+c$
3. $\int cosx\ dx=sinx+c$
4. $\int sec^{2}dx=tanx+c$
5. $\int csc^{2}dx=-cotx+c$
6. $\int cscx\ cotx\ dx=-cscx+c$
7. $\int secx\tan x\ dx=secx+c$
8. $\int e^{x}dx=e^{x}+c$

9. $\int x^r dx = \frac{x^{r+1}}{r+1} + c$

10. $\int a^x dx = \frac{a^x}{\ln a} + c \qquad a \neq 1$

Example 1: find

$$\int 3x^4 dx$$

Sol/

$$\int 3x^4 dx = 3\frac{x^5}{5} + c$$

Example 2: find

$$\int \frac{1}{x^2} dx$$

Sol/

$$\int x^{-2} dx = x^{-1} + c$$

Example 3: find

$$\int x^{\frac{7}{5}} dx$$

Sol/

$$\int x^{\frac{7}{5}} dx = \frac{x^{\frac{12}{5}}}{\frac{12}{5}} + c$$

$$= \frac{5}{12} x^{\frac{12}{5}} + c$$

Example 4: find

$$\int \sqrt[3]{x^2} dx$$

$$\int \sqrt[3]{x^2} dx = \int x^{\frac{2}{3}} dx$$

$$\frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c = \frac{3}{5} x^{\frac{5}{3}} + c$$

Example 5: find $\int (4sec^2x + cscx\ cotx)\ dx$

$$\int (4sec^2x + cscx\ cotx)\ dx = 4tanx - cscx + c$$

Example 6: find

$$\int (\frac{1}{t} - \cos t)\ dt$$

$$\int (\frac{1}{t} - \cos t)\ dt = \ln t - sint + c$$

Examples: evaluate

i. $\int \frac{2x-1}{x^2-x+8} dx$

Sol/

$$\ln(x^2 - x + 8) + c$$

ii. $\int \frac{\cos x}{sin^2x} dx$

Sol/

$$\int \frac{\cos x}{sin\ x} * \frac{1}{\sin x} dx = \int cotx - cscx\ dx$$

$$= -cscx + c$$

iii. $\int 3^x dx$

Sol/

$$\int 3^x dx = \frac{3^x}{\ln 3} + c$$

If $\int \left[f(u) \frac{du}{dx}\right] dx = \int f(u)\ du$

Let F be an antiderivative of f, so that $\frac{d}{du}(F(u)) = f(u)$

$$or \int f(u) du = F(u) + c$$

Examples: evaluate

1. $\int 2x(x^2+1)^{50}dx$

   Sol/

$$\int 2x(x^2+1)^{50}dx = \frac{(x^2+1)^{51}}{51} + c$$

2. $\int sin^2x \cos x \, dx$

   Sol/

$$\int sin^2x \cos x \, dx = \frac{1}{3} sin^3x + c$$

3. $\int \cos 5x \, dx$

   Sol/

$$\frac{1}{5}\int 5\cos 5x \, dx = \frac{1}{5} sin5x + c$$

4. $\int \frac{cos\sqrt{x}}{2\sqrt{x}} dx$

   Sol/

$$\int \frac{cos\sqrt{x}}{2\sqrt{x}} dx = sin\sqrt{x} + c$$

Theorem:

1. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$
2. $\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + c$
3. $\int \frac{dx}{\sqrt{1+x^2}} = \tan^{-1} x \ + c$
4. $\int \frac{-dx}{\sqrt{1+x^2}} = \cot^{-1} x + c$
5. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$
6. $\int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1} x + c$

Examples: Find

1. $\int \left[\frac{4}{x\sqrt{x^{2-1}}} - \frac{1}{\sqrt{1-x^2}}\right] dx$

Sol/

$$\int \left[\frac{4}{x\sqrt{x^{2-1}}} - \frac{1}{\sqrt{1-x^2}}\right] dx = 4\ \sec^{-1} x - \sin^{-1} x + c$$

2. $\int \left[3x - \frac{e^x}{1+e^{2x}}\right] dx$

Sol/

$$\int \left[3x - \frac{e^x}{1+e^{x^2}}\right] dx = \frac{3}{2}x^2 - \tan^{-1} e^x + c$$

Remark:

- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$
- $\int \frac{-dx}{\sqrt{a^2-x^2}} = \cos^{-1}\frac{x}{a} + c$
- $\int \frac{dx}{\sqrt{a^2+x^2}} = \tan^{-1}\frac{x}{a} \ + c$
- $\int \frac{-dx}{\sqrt{a^2+x^2}} = \cot^{-1}\frac{x}{a} + c$
- $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$
- $\int \frac{-dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\csc^{-1}\frac{x}{a} + c$

Example 1: evaluate

1. $\int \frac{dx}{\sqrt{1+3x^2}}$

Sol/

$$\text{let } u = \sqrt{3x} \ \rightarrow du = \sqrt{3}dx \rightarrow dx = \frac{1}{\sqrt{3}}du$$

$$\int \frac{dx}{\sqrt{1+3x^2}} = \int \frac{\frac{1}{\sqrt{3}}dx}{1+u^2}$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{1+u^2}du = \frac{1}{\sqrt{3}}\tan^{-1}u + c$$

$$\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3x}) + c$$

Example 2: evaluate

$$\int \frac{dx}{\sqrt{2-x^2}}$$

Sol/

$$\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

Example 3: evaluate

$$\int \frac{\sin\ x}{cos^2x+1}dx$$

Sol/

$$\text{let } u = cosx \ \rightarrow du = -\sin x\, dx$$

$$\int \frac{\sin\ x}{cos^2x+1}dx = \int \frac{-du}{u^2+1}du$$

$$= -\tan^{-1}u + c \rightarrow = \tan^{-1}(\cos x) + c$$

Exercise: Evaluate the integral

1. $\int \frac{dx}{\sqrt{1-4x^2}}$
2. $\int \frac{dx}{x\sqrt{1-(lnx)^2}}$
3. $\int \frac{dy}{y\sqrt{5y^2-3}}$

The definite integral

Definition: If f is an integral function on [a, b] and F is an antiderivative of f on [a , b] , then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Theorem: If f and g are integral functions on [a, b] and m is constant then,

1. $\int_a^b mf(x)dx = m\int_a^b f(x)dx$
2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
3. $\int_a^b mf(x)dx = -\int_b^a f(x)dx$
4. $\int_a^a f(x)dx = 0$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad where\ c \in (a.b)$

Example1: Evaluate $\int_0^3 (2x^2 - 4x + 1)dx$

Sol/

$$\int_0^3 (2x^2 - 4x + 1)dx = \left[\frac{2}{3}x^3 - \frac{4}{2}x^2 + x\right]_0^3$$

$$= (18 - 18 + 3) - 0 = 3$$

Example 2: Evaluate $\int_0^{\frac{\pi}{4}} \sin(2x) + \cos(4x)\, dx$

Sol/

$$\int_0^{\frac{\pi}{4}} \sin(2x) + \cos(4x)\, dx = \left[\frac{-1}{2}\cos 2x + \frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{4}}$$

$$= (0 + 0) - \left(\frac{-1}{2} + 0\right) = \frac{1}{2}$$

Example 3: Evaluate $\int_0^1 e^x dx$

Sol/

$$\int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = e - 1$$

Example 4: Evaluate

$$\int_{-1}^{2} |x|$$

Sol/

Since $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$$\int_{-1}^{2} |x| = \int_{-1}^{0} -x\, dx + \int_{0}^{2} x\, dx$$

$$= \left[\frac{-x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^2$$

$$= \left(0 + \frac{1}{2}\right) + (2 - 0)$$

$$= \frac{1}{2} + 2$$

$$= \frac{5}{2}$$

Techniques of integration;

(1) **Integration by substitution**

Example 1: Evaluate

$$\int e^{3\cos x}(\sin x)dx$$

Sol/

$$\int e^{3\cos x}(\sin x)dx = \frac{-1}{3}e^{3\cos x} + c$$

Example 2: Evaluate

$$\int_0^2 2x\,(x^2+1)^3 dx$$

Sol/

$$\int_0^2 2x\,(x^2+1)^3 dx = \left[\frac{(x^2+1)^4}{4}\right]_0^2$$

$$= \frac{1}{4}[(x^2+1)^4]_0^2$$

$$= \frac{1}{4}(5^4 - 1)$$

$$= \frac{1}{4}(625 - 1) = 156$$

(2) Integration by part

Theorem: If u = f (x) and v = g (x) then

1. $\int f(x)\, g^{\backslash}(x)\;dx = \int u\,dv = uv - \int v\,du$
2. $\int_a^b (x)\, g^{\backslash}(x)dx = \int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

Example: Evaluate

1. $\int xe^x\,dx$

Sol/

$$\text{let } u = x \rightarrow du = dx$$
$$\text{let } v = e^x \rightarrow dv = e^x$$

$$\int xe^x\,dx = uv - \int v\,du$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

2. $\int \ln x\ dx$

Sol/

$$\text{let } u = \ln x \rightarrow du = \frac{1}{x}dx$$

$$\text{let } v = x \rightarrow dv = dx$$

$$\int \ln x\ dx = uv - \int v\,du$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + c$$

3. $\int_0^1 \tan^{-1} x\ dx$

Sol/

$$\text{let } u = \tan^{-1} x \rightarrow du = \frac{1}{1+x^2}dx$$

$$\text{let } v = x \rightarrow dv = dx$$

$$\int_0^1 \tan^{-1} x\ dx = x\tan^{-1} x - \int_0^1 \frac{x}{1+x^2}\ dx$$

$$= \left[x\tan^{-1} x - \frac{1}{2} ln\ 1 + x^2\right]_0^1$$

$$= (1 * \tan^{-1}(1)) - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

4. $\int_0^{\frac{\pi}{2}} x \sin x\ dx$

Sol/

$$\text{let } u = x \rightarrow du = dx$$

$$\text{let } v = -\cos x \rightarrow dv = \sin x\ dx$$

$$\int_{0}^{\frac{\pi}{2}} x\sin x\,dx = [-x\,cosx]_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} -\cos x\,dx$$

$$= \left[\frac{\pi}{2}\,cos\frac{\pi}{2} - 0\right] + [\sin x]_{0}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} * 0 + sin\frac{\pi}{2} - \sin(0)$$

=1

Theorem:

1. $\int \tan x\,dx = \ln|\sec x| + c$
2. $\int \sec x\,dx = \ln|\sec x + \tan x| + c$
3. $\int \cot x\,dx = -\ln|\csc x| + c$
4. $\int \csc x\,dx = -\ln|\csc x + \cot x| + c$