

## Partial Differential Equations (PDEs):

In problems where one variable, say  $u$ , depends on more than one independent variable, say that  $x$  and  $t$ , then any derivatives of  $u$  will be (partial derivatives) such as  $\frac{\partial u}{\partial x}$  or  $\frac{\partial^2 u}{\partial t^2}$  and any differential equation arising will be known as a partial differential equation.

### ① \* Method of Direct Integration:

Ex(1) Solve;  $\frac{\partial^2 u}{\partial x^2} = x^2 + y$

sol

① Integrate with respect to  $(x)$

$$\int \frac{\partial^2 u}{\partial x^2} = \int (x^2 + y) dx$$

$$\frac{\partial u}{\partial x} = \frac{x^3}{3} + yx + f(y)$$

② Integrate with respect to  $(x)$

$$\int \frac{\partial u}{\partial x} = \int \left( \frac{x^3}{3} + yx + f(y) \right) dx$$

$$\Rightarrow u = \frac{x^4}{12} + y \frac{x^2}{2} + x f(y) + g(y)$$

Example (2) solve  $\frac{\partial^2 Z}{\partial x \partial y} = X^2 y$  that Subject

to the conditions  $Z(x, 0) = x^2$  &  $Z(1, y) = \cos y$   
Sol:

1. Integrate with respect to  $x$ .

$$\frac{\partial Z}{\partial y} = \int X^2 y \, dx \rightarrow \frac{\partial Z}{\partial y} = \frac{1}{3} X^3 y + f(y)$$

2. Integrate with respect to  $y$ .

$$Z = \int \frac{1}{3} X^3 y + f(y) \, dy$$

$$\therefore Z = \frac{1}{6} X^3 y^2 + F(y) + g(x) \quad \text{--- (1)}$$

① Boundary condition  $Z(x, 0) = x^2$

$$Z = x^2 \quad ; \quad y = 0$$

$$\Rightarrow x^2 = \frac{1}{6} X^3 (0)^2 + F(0) + g(x)$$

$$x^2 = F(0) + g(x) \Rightarrow g(x) = x^2 - F(0) \quad \text{] sub. in eq (1)}$$

$$Z = \frac{1}{6} X^3 y^2 + F(y) + x^2 - F(0) \quad \text{--- (2)}$$

② Boundary condition  $Z = \cos y$  ;  $X = 1$

$$\cos y = \frac{1}{6} (1)^3 y^2 + F(y) + 1^2 - F(0)$$

$$F(y) = \cos y - \frac{1}{6} y^2 - 1 + F(0) \quad ] \text{ sub in eq ②}$$

$$Z = \frac{1}{6} X^3 y^2 + \cos y - \frac{1}{6} y^2 - X + F(0) + X - F(0)$$

$$Z = \frac{1}{6} X^3 y^2 + \cos y - \frac{1}{6} y^2$$

Example (3) Solve  $(X^3 + 3X + 4) \frac{dy}{dx} = X^2 + 1$

Sol:

$$\frac{dy}{dx} = \frac{X^2 + 1}{X^3 + 3X + 4}$$

$$dy = \frac{X^2 + 1}{X^3 + 3X + 4} dx \quad ] \text{ Integrate both sides}$$

$$\int dy = \frac{1}{3} \int \frac{3(X^2 + 1)}{X^3 + 3X + 4} dx$$

$$y = \frac{1}{3} \ln(X^3 + 3X + 4) + F(y)$$

Example (4) Solve  $\frac{\partial^3 z}{\partial x^2 \partial y} = 18x^2 y^2 + \sin(2x-y)$

Sol: Integrate w.r. to x

$$\frac{\partial^2 z}{\partial x \partial y} = 18 \frac{x^2}{2} y^2 - \frac{\cos(2x-y)}{2} + f(y)$$

Integrate w.r. to x

$$\frac{\partial z}{\partial y} = 9 \frac{x^3}{3} y^2 - \frac{1}{2} \frac{\sin(2x-y)}{2} + F(y)$$

Integrate w.r. to y

$$z = 3x^3 \frac{y^3}{3} - \frac{1}{4} [-\cos(2x-y)] + F(y) + g(x)$$

$$\therefore z = x^3 y^3 - \frac{1}{4} \cos(2x-y) + F(y) + g(x)$$