Ministry of Higher Education and Scientific Research

Al- Mustaqbal University

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جامعة المستقبل
AL MUSTAQBAL UNIVERSITY

Mathematics IV

2nd Stage / 2nd Semester

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2024 – 2023

First Order Differential Equations F.O.D.E

① Variable separation:

This form is $\int f(x)\,dx + \int g(y)\,dy = C$

Ex(1) Solve; $dx + xy\,dy = y^2 dx + y\,dy$?

sol:

$$(1-y^2)\,dx + y(x-1)\,dy = 0 \;\Big]\; \div (x-1)(1-y^2)$$

$$\frac{dx}{x-1} + \frac{y\,dy}{1-y^2} = 0 \;\Big]\; \text{by Integration}$$

$$\ln|x-1| - \frac{1}{2}\ln|1-y^2| = C \;\Big]\; *2$$

$$2\ln|x-1| - \ln|1-y^2| = 2C$$

$$\Rightarrow \ln\left|\frac{(x-1)^2}{1-y^2}\right| = 2C \;\Big]\; \text{take exp.}$$

$$\frac{(x-1)^2}{1-y^2} = K$$

Ex(2) solve $\dfrac{dy}{dx} = \dfrac{x^2}{y}$?

sol:

$$y\,dy = x^2\,dx \;\Big]\; \text{Integrate}$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C \Rightarrow y^2 = \frac{2}{3}x^3 + C$$

$$\therefore y = \mp\sqrt{\frac{2}{3}x^3 + C}$$

(2) Homogenous: it can be solved by introducting a new dependent variable:

$\rightarrow y = u x \Longrightarrow dy = u\,dx + x\,du$

Ex(1) Solve $(x^2 + 3y^2)dx = 2xy\,dy$?

Sol:

$y = ux$ and $dy = u\,dx + x\,du$] sub in D.E.

$(x^2 + 3u^2x^2)dx = 2ux^2(u\,dx + x\,du)$

$(x^2 + 3u^2x^2)dx = 2u^2x^2dx + 2ux^3du$

$(x^2 + u^2x^2)dx = 2ux^3du$

$x^2(1+u^2)dx = 2u\,x^3du$] $\div x^3(1+u^2)$

$\int \frac{1}{x}dx = \int \frac{2u}{1+u^2}du$] Integrate

$\ln|x| = \ln|1+u^2| + c$

$\ln|x| - \ln|1+u^2| = c$

$\ln\left|\frac{x}{1+u^2}\right| = c$] by exp.

$\frac{x}{1+u^2} = k \qquad (y = ux \rightarrow u = \frac{y}{x})$

$\Longrightarrow \frac{x}{1+(\frac{y}{x})^2} = k$

Ex(2) Solve $(x^3+y^3)\,dx - 3xy^2\,dy = 0$?

sol:

$$y = ux \longrightarrow dy = u\,dx + x\,du \;\Big]\; \text{sub in D.E.}$$

$$(x^3 + u^3x^3)\,dx - 3x^3u\,(u\,dx + x\,du) = 0$$

$$(x^3 + u^3x^3)\,dx - 3x^3u^3\,dx - 3x^4u^2\,du = 0$$

$$(x^3 - 2u^3x^3)\,dx - 3x^4u^2\,du = 0$$

$$x^3(1-2u^3)\,dx = 3x^4u^2\,du \;\Big]\; \div\, x^4(1-2u^3)$$

$$\int \frac{1}{x}\,dx = \frac{-1}{2}\int \frac{3u^2}{1-2u^3}\,du$$

$$\ln|x| = \ln|1-2u^3|^{-\frac{1}{2}} + c \;\Big]\; \text{take exp.}$$

$$x = \frac{1}{\sqrt{1-2u^3}} + e^c \qquad [e^c = k]$$

$$\Rightarrow x = \frac{1}{\sqrt{1-2\left(\frac{y}{x}\right)^3}} + k \qquad \left[y = ux \rightarrow u = \frac{y}{x}\right]$$

③ Exact

standard form: $m(x,y)\,dx + N(x,y)\,dy = 0$ ...... ①

eq.1 is exact if $\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$

To solve the exact eq. we first solve

$m = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$

EX.(1) Solve $(4x^3y^3 - 2xy) + (3x^4y^2 - x^2)dy$?

sol:

$m = 4x^3y^3 - 2xy \rightarrow \frac{\partial m}{\partial y} = 12x^3y^2 - 2x$

$N = 3x^4y^2 - x^2 \rightarrow \frac{\partial N}{\partial x} = 12x^3y^2 - 2x$

∴ Exact equation

$m = \frac{\partial u}{\partial x}$

$4x^3y^3 - 2xy = \frac{\partial u}{\partial x}$

$\int \partial u = \int (4x^3y^3 - 2xy)\,dx$

$\Rightarrow u = x^4y^3 - x^2y + g(y)$

$N = \frac{\partial u}{\partial y}$

$3x^4y^2 - x^2 = \frac{\partial(x^4y^3 - x^2y + g(y))}{\partial y}$

$3x^4y^2 - x^2 = 3x^4y^2 - x^2 + g'(y)$

$g'(y) = 0 \implies g(y) = C$

$\therefore u = x^4y^3 - x^2y + C$

Ex.(2) Solve $(3e^{3x}y - 2x)dx + e^{3x}dy$?

sol:

$m = 3e^{3x}y - 2x \longrightarrow \frac{\partial m}{\partial y} = 3e^{3x}$

$N = e^{3x} \longrightarrow \frac{\partial N}{\partial x} = 3e^{3x}$

$\therefore$ Exact equation

$m = \frac{\partial u}{\partial x}$

$3e^{3x}y - 2x = \frac{\partial u}{\partial x} \Longrightarrow \int \partial u = \int (3e^{3x}y - 2x)dx$

$u = e^{3x}y - x^2 + g(y)$

$N = \frac{\partial u}{\partial y}$

$e^{3x} = \frac{\partial (e^{3x}y - x^2 + g(y))}{\partial y} \Longrightarrow e^{3x} = e^{3x} + g'(y)$

$\Longrightarrow g'(y) = 0$

$\therefore g(y) = C$

$\Longrightarrow u = e^{3x}y - x^2 + C$

④ Linear D.E

General form: $\frac{dy}{dx} + p(x)\,y = Q(x)$; to solve it :-

① Find integration factor $I.f. = e^{\int p(x)\,dx}$

② Integrate and solve $y(x) = \frac{1}{I.f.}\int I.f. * Q(x)\,dx + \frac{c}{I.f.}$

Ex(1) Solve $x\frac{dy}{dx} + (1-x)y = xe^{x}$?

$\frac{dy}{dx} + (\frac{1}{x} + 1)y = e^{x}$ [linear form]

$I.f = e^{\int(\frac{1}{x}+1)dx} \Rightarrow = e^{\ln x - x} \Rightarrow = xe^{-x}$

$y(x) = \frac{1}{xe^{-x}}\int xe^{-x} * e^{x}\,dx + \frac{c}{xe^{-x}}$

$= \frac{e^{x}}{\cancel{x}} * \frac{x^{2}}{2} + \frac{c}{xe^{-x}}$

$\therefore y(x) = \frac{xe^{x}}{2} + \frac{ce^{x}}{x}$

EX.(2) Solve $\frac{dy}{dx} + y\cot x = \sin 2x$?

sol:

$$I.f = e^{\int \cot x\, dx} = e^{\int \frac{\cos x}{\sin x} dx} = \cancel{e}^{\cancel{\ln}\sin x}$$

$\therefore I.f = \sin x$

$$y(x) = \frac{1}{\sin x}\int \sin x * \sin 2x\, dx + \frac{c}{\sin x}$$

$$= \frac{1}{\sin x}\int \sin x * 2\sin x \cos x\, dx + \frac{c}{\sin x}$$

$$= \frac{2}{\sin x}\int \sin^2 x \cos x\, dx + \frac{c}{\sin x}$$

$$= \frac{2}{\cancel{\sin x}} * \frac{\sin^{\cancel{3}\,2} x}{3} + \frac{c}{\sin x}$$

$$y(x) = \frac{2}{3}\sin^2 x + \frac{c}{\sin x}$$

5 Bernoulli's equation

standard form: $\frac{dy}{dx} + p(x)\,y = Q(x)\,y^n$

to solve it, we set $z = y^{1-n}$

Ex. Solve $\frac{dy}{dx} + \frac{1}{x}y = x\,y^2$?

Sol: $z = y^{1-2} \rightarrow z = y^{-1} \rightarrow z = \frac{1}{y} \rightarrow y = \frac{1}{z}$ } sub in D.E.

$dy = \frac{-1}{z^2}\,dz$

$\left.\frac{-1}{z^2}\frac{dz}{dx} + \frac{1}{xz} = x\frac{1}{z^2}\right] * (-z^2)$

$\frac{dz}{dx} - \frac{z}{x} = -x$ (linear form)

I.f. $e^{\int \frac{-1}{x}dx} \rightarrow = e^{-\ln x} \Rightarrow$ I.f $= \frac{1}{x}$

$z(x) = x\int \frac{1}{\cancel{x}}\cdot(-\cancel{x})\,dx + cx$

$z(x) = -x^2 + cx$

$\frac{1}{y} = -x^2 + cx$

$\therefore y(x) = \frac{1}{-x^2 + cx}$