Second Order Differential Equations (S.O.D.E)

1. Non-linear SODEs: non linear SODEs may be classified as:
   (A) Equations with dependent Variable missing.
   (B) Equations with independent variable missing.
   (C) Homogenous equations.

* Classification of equations:

Ex/ $\frac{d^2y}{dx^2} + 2\sin y = 0$

→ non linear D.E because the dependent variable apear as $\sin y$, $\cos y$, $\tan y$, e^y

Ex.2/ $\frac{d^2y}{dx^2} + 4y\frac{dy}{dx} + 2y = \cos x$

→ non linear D.E because $y\frac{dy}{dx}$

Ex.3/ $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

→ non linear D.E because (y) is missing and $x\frac{d^2y}{dx^2}$

(A) Equations with dependent variable missing:

General formula: $f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$

Let: $p = \frac{dy}{dx} \longrightarrow \frac{dp}{dx} = \frac{d^2y}{dx^2}$

eleminate y and reduce the equation to 1st order.

(B) Equations with independent variable missing:

General formula: $f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$

let: $p = \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$$= \frac{dp}{dy} \cdot \frac{dy}{dx} \longrightarrow = p \frac{dp}{dy}$$

$$\therefore \frac{d^2y}{dx^2} = p \frac{dp}{dy}$$

EX1: Solve the D.E. $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$ ...... (1)

Sol:

Let $p=\frac{dy}{dx} \rightarrow \frac{dp}{dx}=\frac{d^2y}{dx^2} \rightarrow$ sub. in eq (1)

$x\frac{dp}{dx}+p=0 \rightarrow x\frac{dp}{dx}=-p$

$\frac{dp}{p}=-\frac{dx}{x}$ [Integration]

$\ln p=-\ln x+\ln c \rightarrow p=\frac{c}{x}=\frac{dy}{dx}$

$\rightarrow \frac{c}{x}dx=dy$ [Integration]

$y=c\ln x+K$

EX (2) Solve; $y\frac{d^2y}{dx^2}+1=\left(\frac{dy}{dx}\right)^2$ ------ (1)

Sol:

Let $p=\frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2} = p\frac{dp}{dy}$ sub. in eq. 1

$y\,p\frac{dp}{dy}+1=p^2 \longrightarrow \frac{p}{p^2-1}\,dp=\frac{1}{y}\,dy$

by Integration:

$\ln y=\frac{1}{2}\ln(p^2-1)+\ln c$

$y=c\,(p^2-1)^{\frac{1}{2}}$

$p=\sqrt{\frac{y^2}{c^2}+1}$ $p=\frac{dy}{dx}$

$\frac{dy}{dx}=\sqrt{\frac{y^2}{c^2}+1}$ (separate)

$\int dx=\int\frac{dy}{\sqrt{\frac{y^2}{c^2}+1}}$

$x=\sinh^{-1}\left(\frac{y}{c}\right)+b$

EX. 3 Solve $y'' + 3y^2 y' = 0$ that Subjected to the Conditions: $y(1) = 2$ and $y'(1) = 1$.

Solution:

let $p = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$ (Sub in eq.)

$p\frac{dp}{dy} + 3y^2 p = 0 \rightarrow p\frac{dp}{dy} = -3y^2 p$] $\div p$

$\int dp = \int -3y^2 dy \Rightarrow p = -y^3 + c$

$\frac{dy}{dx} = -y^3 + c$ Sub: $y'(1) = 1$

$1 = -y^3 + c \rightarrow c = 1 + y^3$

$\frac{dy}{dx} = -y^3 + 1 + y^3 \rightarrow \frac{dy}{dx} = 1$

$\int dy = \int 1\,dx$

$y = x + c_1$ Sub $y(1) = 2$

$2 = 1 + c_1 \rightarrow c_1 = 1$

$\therefore y = x + 1$

EX. 4: Solve the D.E $xy'' + y' = 4x$ ——— ①

Solution:

let $y' = p \rightarrow y'' = \frac{dp}{dx}$ [sub in eq. 1]

$x\frac{dp}{dx} + p = 4x$

$x\frac{dp}{dx} + p = \frac{d}{dx}(px)$

$\int \frac{d}{dx}(px) = \int 4x$

$px = \frac{4}{2}x^2 + c_1 \longrightarrow p = 2x + \frac{c_1}{x}$

$\int \frac{dy}{dx} = \int 2x + \frac{c_1}{x}$

$\therefore y = x^2 + c_1 \ln x + c_2$

EX 5: Solve the D.E $2yy'' = 1 + (y')^2$ ---- ①

Solution:

Let $y' = p \rightarrow y'' = p\frac{dp}{dx}$ sub in eq. 1

$$2yp\frac{dp}{dy} = 1 + p^2 \rightarrow \frac{dy}{y} = \frac{2p}{1+p^2}\,dp$$

$$\ln(1+p^2) = \ln y + \ln C_1$$

$$\ln(1+p^2) = \ln yC_1 \rightarrow 1+p^2 = yC_1$$

$$p = \sqrt{yC_1 - 1} \rightarrow \frac{dy}{dx} = (yC_1 - 1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{(yC_1-1)^{\frac{1}{2}}} = dx$$

$$\int (yC_1 - 1)^{\frac{-1}{2}}\,dy = \int dx$$

$$\frac{2}{C_1}(C_1 y - 1)^{\frac{1}{2}} = x + C_2$$