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Lecture Six

Sinusoidal Alternating Waveforms

6.1 Introduction

Each waveform of Fig. 6.1 is an **alternating waveform** available from commercial supplies. The term alternating indicates only that the waveform alternates between two prescribed levels in a set time sequence (Fig. 6.1).

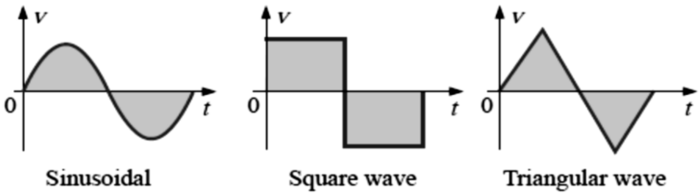

Fig. 6.1 Alternating waveforms.

To be absolutely correct, the term sinusoidal, square wave, or triangular must also be applied.

6.2 Sinusoidal AC Voltage Characteristics and Definitions

Definitions

The sinusoidal waveform of Fig. 6.2 with its additional notation will now be used as a model in defining a few basic terms.

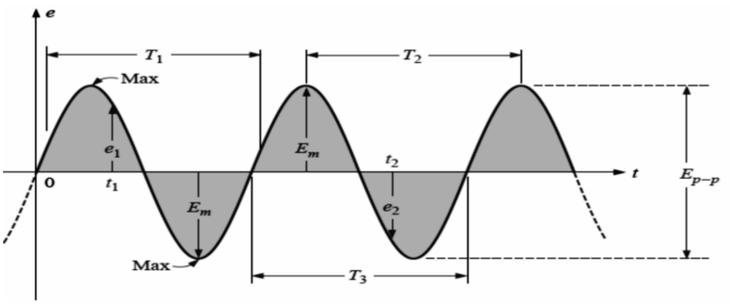

Fig. 6.2 Important parameters for a sinusoidal voltage.

Waveform: The path traced by a quantity, such as the voltage in Fig. 6.2, plotted as a function of some variable such as time (as above), position, degrees, radians, temperature, and so on.

Instantaneous value: The magnitude of a waveform at any instant of time; denoted by lowercase letters (**e_1**, **e_2**).

Peak amplitude: The maximum value of a waveform as measured from its average, or mean, value, denoted by uppercase letters (such as $\mathbf{E_m}$ for sources of voltage and $\mathbf{V_m}$ for the voltage drop across a load).

Peak value: The maximum instantaneous value of a function as measured from the zero-volt level. For the waveform of Fig. 6.2, the peak amplitude and peak value are the same, since the average value of the function is zero volts.

Peak-to-peak value: Denoted by E_{p-p}or V_{p-p}, the full voltage between positive and negative peaks of the waveform, that is, the sum of the magnitude of the positive and negative peaks.

Periodic waveform: A waveform that continually repeats itself after the same time interval. The waveform of Fig. 6.2 is a periodic waveform.

Period (*T*): The time interval between successive repetitions of a periodic waveform (the period $T_1 = T_2 = T_3$ in Fig. 6.2).

Cycle: The portion of a waveform contained in one period of time. The cycles within T_1, T_2, and T_3 of Fig. 6.2 may appear different in Fig. 6.3.

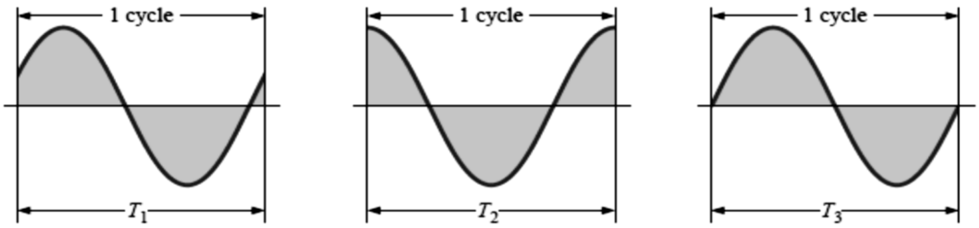

Fig. 6.3 Defining the cycle and period of a sinusoidal waveform.

Frequency (*f*): The number of cycles that occur in 1 s. The frequency of the waveform of Fig. 6.4(a) is 1 cycle per second, and for Fig. 6.4(b), 2.5 cycles per second, and while for Fig. 6.4(c) the frequency will be 2 cycles per second.

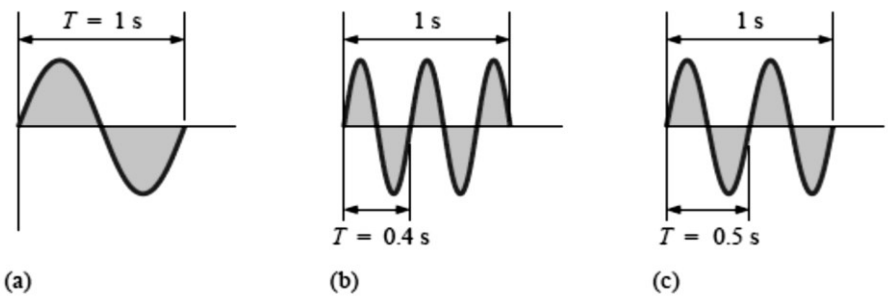

Fig. 6.4 Demonstrating the effect of a changing frequency on the period of a sinusoidal waveform.

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The unit of measure for frequency is the hertz (H_z),

where

1 hertz (H_z) = 1 cycle per second (c/s) (6.1)

Since the frequency is inversely related to the period—that is, as one increases, the other decreases by an equal amount—the two can be related by the following equation:

$$\mathbf{f} = \frac{\mathbf{1}}{\mathbf{T}} \tag{6.2}$$

where f in hertz (H_z) and T in seconds (s)

or $$\mathbf{T} = \frac{\mathbf{1}}{\mathbf{f}} \tag{6.3}$$

Example 6.1: Find the period of a periodic waveform with a frequency of

a. 60 H_z;

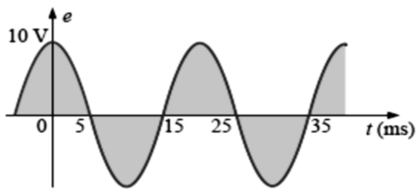
b. 1000 H_z.

Solution:

a. $\mathbf{T} = \frac{1}{f} = \frac{1}{6Hz} = \mathbf{0.01667}s\ or\ \mathbf{16.67}\ ms$

b. $\mathbf{T} = \frac{1}{f} = \frac{1}{1000Hz} = \mathbf{10^{-3}}s = \mathbf{1}\ ms$

Example 6.2: Determine the frequency of the waveform shown below.



Solution: From the figure, T = (25 ms - 5 ms) = 20 ms, and

$$\mathbf{f} = \frac{\mathbf{1}}{\mathbf{T}} = \frac{\mathbf{1}}{\mathbf{20\times10^{-3}s}} = \mathbf{50}\ Hz$$

6.3 The Sine Wave

The sinusoidal waveform is the only alternating waveform whose shape is unaffected by the response characteristics of R, L, and C elements.

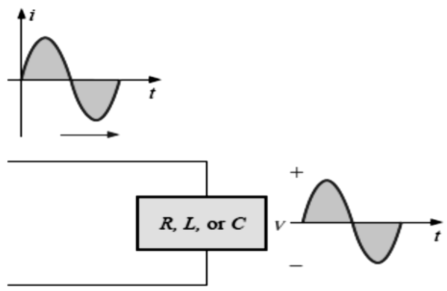


Fig. 6.5

The unit of measurement for the angle axis of is the ***degree***.

A second unit of measurement frequently used is the ***radian*** (***rad***).

π in rad = 180°

2π in rad = 360° (6.4)

With 1 rad = 57.296° ≅ 57.3° (6.5)

The quantity π is the ratio of the circumference of a circle to its diameter.

For 180° and 360°, the two units of measurement are related.

The conversion equations between the two are the following:

$$\text{Radians} = \left(\frac{\pi}{180^\circ}\right) \times (\text{degrees}) \quad (6.6)$$

$$\text{Degrees} = \left(\frac{180^\circ}{\pi}\right) \times (\text{radians}) \quad (6.7)$$

The velocity with which the radius vector rotates about the center, called the angular velocity, can be determined from the following equation:

$$\textbf{Angular velocity} = \frac{\textit{distance (degrees or radians)}}{\textit{time (seconds)}} \quad (6.8)$$

Substituting into Eq. (6.8) and assigning the Greek letter omega (ω) to the angular velocity, we have

$$\omega = \alpha/t \quad \Longrightarrow \alpha = \omega t \quad (6.9)$$

Since ω is typically provided in radians per second, the angle α obtained using Eq. (6.9) is usually in radians.

The angular velocity ω of the rotating radius vector is

$$\boldsymbol{\omega = 2\pi f} \quad (6.10)$$

Example 6.3: Determine the angular velocity of a sine wave having a frequency of 60 Hz.

Solution: $\omega = 2\pi f = (2\pi)(60\ \text{Hz}) = 377\ \text{rad/s}$

Example 6.4: Given ω= 200 rad/s, determine how long it will take the sinusoidal waveform to pass through an angle of 90°.

Solution: Eq. (6.9): α =ω t, and t = α /ω However, a must be substituted as π/2 (= 90°)

$$t = \alpha/\omega = \frac{\pi/2}{200} = \mathbf{7.85\ ms}$$

Example 6.5: Find the angle through which a sinusoidal waveform of 60 Hz will pass in a period of 5 ms.

Solution: Eq. (6.9): α=ω t, or α= 2π ft = (2π)(60 Hz)(5×10^{-3} s) = 1.885 rad = 108°

6.4 General Format for the Sinusoidal Voltage or Current

The basic mathematical format for the sinusoidal waveform is.

$$\mathbf{A_m \sin \alpha} \quad (6.11)$$

where A_m is the peak value of the waveform and **α** is the unit of measure for the horizontal axis, as shown in Fig. 6.6.

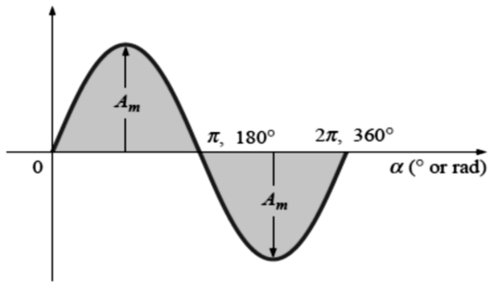


Fig. 6.6 Basic sinusoidal function.

Due to Eq. (6.9), the general format of a sine wave can also be written

$$\mathbf{A_m \sin \omega t} \quad (6.12)$$

For electrical quantities such as current and voltage, the general format is

$$\mathbf{i = I_m \sin \omega t = I_m \sin \alpha}$$

$$\mathbf{e = E_m \sin \omega t = E_m \sin \alpha}$$

where the capital letters with the subscript m represent the amplitude, and the lowercase letters i and e represent the instantaneous value of current or voltage, respectively, at any time t.

Example 6.6: Given e = 5 sin α, determine e at α= 40° and α= 0.8π.

Solution: For α = 40°

e = 5 sin 40° = 5(0.6428) = 3.214 V

For α= 0.8π,

α (°) =(180/π)(0.8π) = 144°

and e = 5 sin 144° = 5(0.5878) = 2.939 V

6.5 Phase Relations

Thus far, we have considered only sine waves that have maxima at π/2 and 3π/2, with a zero value at 0, π, and 2π. If the waveform is shifted to the right or left of 0°, the expression becomes

$$\mathbf{A_m \sin(\omega t \pm \theta)} \tag{6.13}$$

where θ is the angle in degrees or radians that the waveform has been shifted.

If the waveform passes through the horizontal axis with a positive- going (increasing with time) slope before 0°, as shown in Fig. 6.7, the expression is

$$\mathbf{A_m \sin(\omega t + \theta)} \tag{6.14}$$

At ωt =α= 0°, the magnitude is determined by $A_m \sin\theta$.

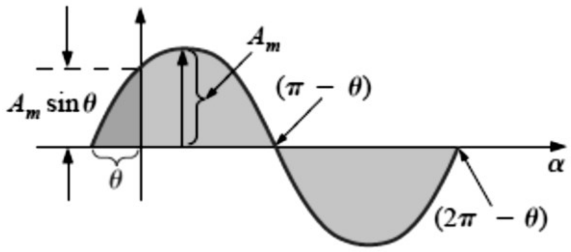


Fig. 6.7 Defining the phase shift for a sinusoidal function that crosses the horizontal axis with a positive slope before 0°.

If the waveform passes through the horizontal axis with a positive-going slope after 0°, as shown in Fig. 6.8, the expression is

$$\mathbf{A_m \sin(\omega t - \theta)} \tag{6.15}$$

at ωt =α= 0°, the magnitude is $A_m \sin(-\theta)$, which, by a trigonometric identity, is $-A_m \sin\theta$.

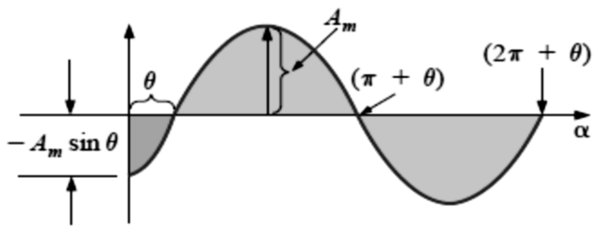


Fig. 6.8 Defining the phase shift for a sinusoidal function that crosses the horizontal axis with a positive slope after 0°.

The geometric relationship between various forms of the sine and cosine functions can be listed below:

$\cos\alpha = \sin(\alpha + 90°)$, $\sin\alpha = \cos(\alpha - 90°)$

$-\sin\alpha = \sin(\alpha \pm 180°)$, $-\cos\alpha = \sin(\alpha \pm 270°) = \sin(\alpha - 90°)$ etc.

$\sin(-\alpha) = -\sin\alpha$, $\cos(-\alpha) = \cos\alpha$

Example 6.7: What is the phase relationship between the sinusoidal waveforms of each of the following sets?

a. $v = 10\sin(\omega t + 30°)$, $i = 5\sin(\omega t + 70°)$

b. $i = 15\sin(\omega t + 60°)$, $v = 10\sin(\omega t - 20°)$

c. $i = 2\cos(\omega t + 10°)$, $v = 3\sin(\omega t - 10°)$

d. $i = -\sin(\omega t + 30°)$, $v = 2\sin(\omega t + 10°)$

e. $i = -2\cos(\omega t - 60°)$, $v = 3\sin(\omega t - 150°)$

Solution:

a. See Fig. 6.9.

i leads v by 40°, or v lags i by 40°.

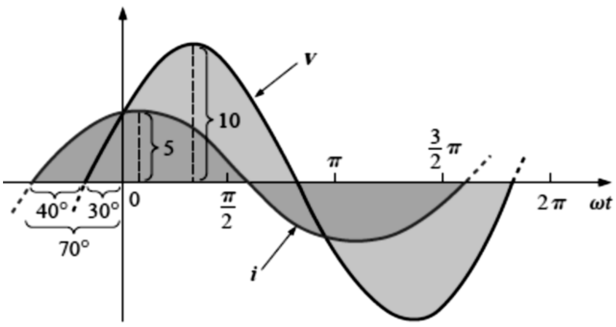


Fig. 6.9 i leads v by 40°.

b. See Fig. 6.10.

i leads v by 80°, or v lags i by 80°.

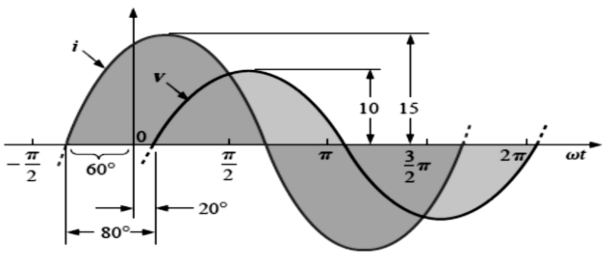


Fig. 6.10 i leads v by 80°.

c. See Fig. 6.11.

$i = 2\cos(\omega t + 10°) = 2\sin(\omega t + 10° + 90°)$

$= 2\sin(\omega t + 100°)$

i leads v by 110°, or v lags i by 110°.

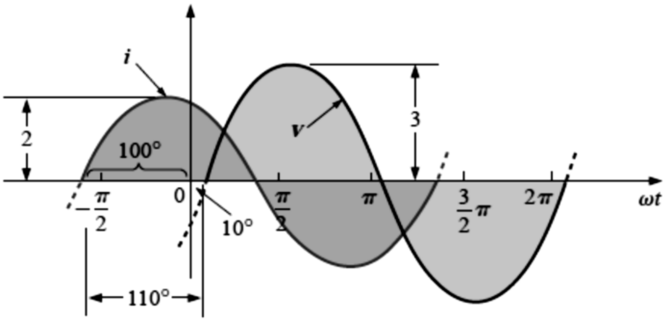


Fig. 6.11 i leads v by 110°.

d. See Fig. 6.12.

Note

$-\sin(\omega t + 30°) = \sin(\omega t + 30° - 180°)$

$= \sin(\omega t - 150°)$

v leads i by 160°, or i lags v by 160°.

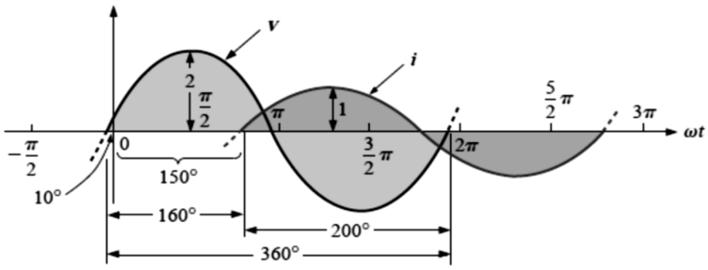


Fig. 6.12 v leads i by 160°.

e. See Fig. 6.13.

$i = -2\cos(\omega t - 60°) = 2\cos(\omega t - 60° - 180°)$

$= 2\cos(\omega t - 240°)$

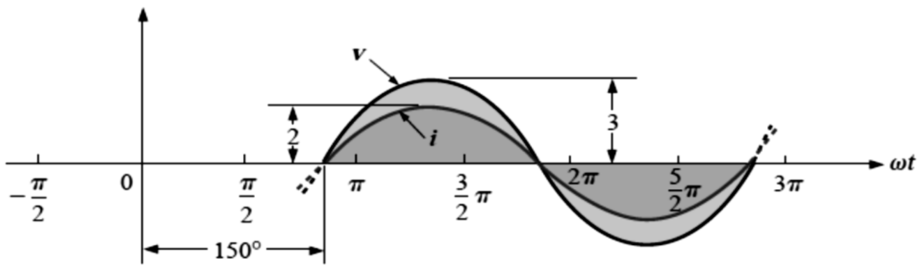


Fig. 6.13 v and i are in phase.

6.6 Effective Root-Mean-Square (R.M.S.) Value

The R.M.S. value of an alternating current is given by that steady (D.C.) current which when flowing through a given circuit for a given time produces the same heat as produced by the alternating current when flowing through the same circuit for the same time. It is also known as the effective or virtual value of the alternating current, the former term being used more extensively.

For computing the R.M.S. value of symmetrical sinusoidal alternating currents, either mid-ordinate method or analytical method may be used, although for symmetrical but non- sinusoidal waves, the mid- ordinate method would be found more convenient.

6.6.1 Mid-ordinate Method

In Fig. 6.14 are shown the positive half cycles for both symmetrical sinusoidal and non-sinusoidal alternating currents. Divide time base 't' into **n** equal intervals of time each of duration t/n seconds.

Let the average values of instantaneous currents during these intervals be respectively $i_1, i_2, i_3, \ldots i_n$ (i.e. mid-ordinates in Fig. 6.14).

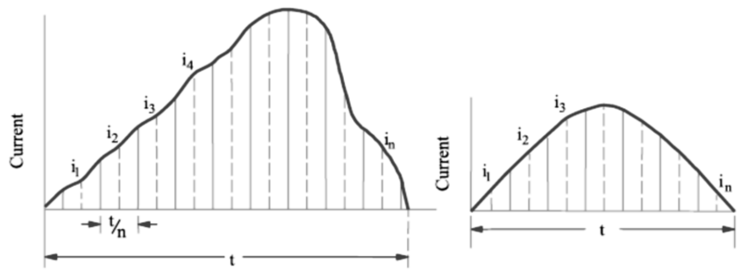


Fig. 6.14

The R.M.S. value of alternating current is

$$I^2 = \frac{i_1^2 + i_2^2 + \cdots + i_n^2}{n}$$

$I = \sqrt{\frac{i_1^2+i_2^2+\cdots+i_n^2}{n}}$= square root of the mean of the squares of the instantaneous currents

Similarly, the R.M.S. value of alternating voltage is given by the expression

$$V = \sqrt{\frac{v_1^2 + v_2^2 + \cdots + v_n^2}{n}}$$

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6.6.2 Analytical Method

The general form of R.M.S. value is

$$I^2 = \frac{1}{T}\int_0^T i^2 dt \qquad \Rightarrow \qquad I = \sqrt{\frac{\int_0^T i^2 dt}{T}} \; or \; I = \sqrt{\frac{\int_0^{2\pi} i^2 d\theta}{(2\pi - 0)}}$$

The standard form of a sinusoidal alternating current is i = I_m sin ωt = I_m sin θ. The mean of the squares of the instantaneous values of current over one complete cycle is (even the value over half a cycle will do).

$$I = \sqrt{\frac{\int_0^{2\pi} i^2 d\theta}{(2\pi - 0)}} = \sqrt{\frac{I_m^2}{2}}$$

The square root of this value is

$$I = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Hence, we find that for a symmetrical sinusoidal current

R.M.S. value of current = 0.707 × max. value of current

The R.M.S. value of an alternating current is of considerable importance in practice, because the ammeters and voltmeters record the R.M.S. value of alternating current and voltage respectively. In electrical engineering work, *unless indicated otherwise, the values of the given current and voltage are always the* R.M.S. *values.*

It should be noted that the average heating effect produced during one cycle is

$$= I^2 R = \left(\frac{I_m}{\sqrt{2}}\right)^2 R = \frac{1}{2} I_m^2 R$$

6.7 Average Value

The average value **I_a** of an alternating current is expressed *by that steady current which transfers across any circuit the same charge as is transferred by that alternating current during the same time*. In the case of a symmetrical alternating current (i.e. one whose two half-cycles are exactly similar, whether sinusoidal or non-sinusoidal), the average value over a complete cycle is zero.

(i) Mid-ordinate Method

With reference to Fig. 6.14, $\boldsymbol{I_{av}} = \frac{\boldsymbol{i_1 + i_2 + \cdots + i_n}}{\boldsymbol{n}}$

This method may be used both for sinusoidal and non-sinusoidal waves, although it is specially convenient for the latter.

(ii) Analytical Method

The general form of average value is

$$I_{av} = \frac{1}{T}\int_0^T i dt, \qquad or\ I_{av} = \frac{1}{2\pi}\int_0^{2\pi} i\, d\theta$$

The standard equation of an alternating current is, i = I_m sin θ

$$I_{av} = \frac{1}{\pi}\int_0^{\pi} I_m sin\theta\, d\theta$$

∴ $I_{av} = 0.637\, I_m$

∴ average value of current = 0.637 × maximum value

Note: The R.M.S. value is always greater than average value except in the case of a rectangular wave when both are equal.

Example 6.8: Compute the average and effective values of the square voltage wave shown in Fig. 6.15.

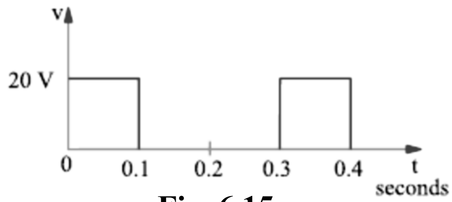

Fig. 6.15

Solution: As seen, for 0 < t < 0.1 i.e. for the time interval 0 to 0.1 second, v = 20 V. Similarly, for 0.1 < t < 0.3, v = 0.

Also time-period of the voltage wave is 0.3 second.

$$\therefore V_{av} = \frac{1}{T}\int_0^T vdt = \frac{1}{0.3}\int_0^{0.1} 20dt$$

$$= \frac{1}{0.3}(20 \times 0.1) = 6.6667\, V$$

$$V = \sqrt{\frac{\int_0^T v^2 dt}{T}} = \sqrt{\frac{\int_0^{0.1} 20^2 dt}{0.3}} = \sqrt{\frac{400 \times 0.1}{0.3}} = 11.5\, V$$

Example 6.9: Calculate the R.M.S. value of the function shown in Fig. 6.16 if it is given that for $0 < t < 0.1$, $y = 10(1 - e^{-100t})$ and $0.1 < t < 0.2$, $y = 10\ e^{-50(t-0.1)}$

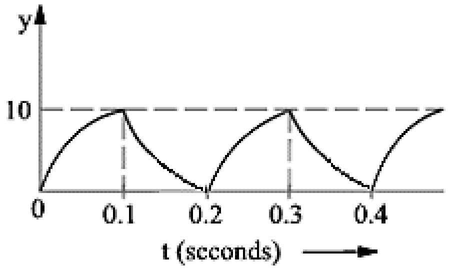

Fig. 6.16

Solution:

$$Y^2 = \frac{1}{0.2}\left\{\int_0^{0.1} y^2 dt + \int_{0.1}^{0.2} y^2 dt\right\}$$

$$= \frac{1}{0.2}\left\{\int_0^{0.1} 10^2(1 - e^{-100t})^2 dt + \int_{0.1}^{0.2} 10^2\left(e^{-50(t-0.1)}\right)^2 dt\right\}$$

$$= 500 \times 0.095 = 47.5 \therefore Y = \sqrt{47.5} = 6.9$$

Example 6.10: Determine the R.M.S. and average value of the waveform shown in Fig. 6.17?

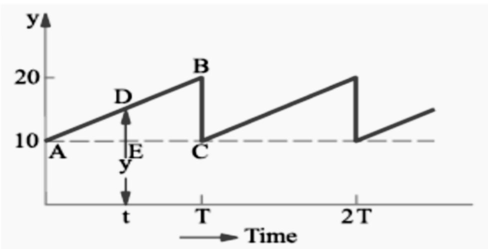

Fig. 6.17

Solution:

The slope of the curve AB is

BC/AC = 10/T.

Next, consider the function y at any time t. It is seen that

y = 10 + (10/T)t

This gives us the equation for the function for one cycle.

$$\boldsymbol{Y_{av} = \frac{1}{T}\int_0^T y dt = \frac{1}{T}\int_0^T \left(10 + \frac{10t}{T}\right) dt} = 15$$

Mean square value $= \frac{1}{T}\int_0^T y^2 dt = \frac{1}{2}\int_0^T \left(10 + \frac{10}{T}t\right)^2 dt = \frac{700}{3}$

or R.M.S. value $= 10\sqrt{7/3} = 15.2$