Lecture Seven

Phasors and The Basic Elements

7.1 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number z can be written in rectangular form as

$$\mathbf{z = x + jy} \tag{7.1a}$$

where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z.

The complex number z can also be written in polar or exponential form as

$$\mathbf{z = r \angle \varphi = re^{j\varphi}} \tag{7.1b}$$

where r is the magnitude of z, and φ is the phase of z. We notice that z can be represented in three ways:

$\mathbf{z = x + jy}$ Rectangular form

$\mathbf{z = r \angle \varphi}$ Polar form

$\mathbf{z = r\, e^{j\varphi}}$ Exponential form (7.2)

The relationship between the rectangular form and the polar form is shown below, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and φ as

$$r = \sqrt{x^2 + y^2}\ , \ \varphi = \tan^{-1}\frac{y}{x} \tag{7.3a}$$

On the other hand, if we know r and φ, we can obtain x and y as

$$\mathbf{x = r \cos \varphi,\ \ y = r \sin \varphi} \tag{7.3b}$$

Thus, z may be written as

$$\mathbf{z = x + jy = r \angle \varphi = r\,(\cos \varphi + j \sin \varphi)} \tag{7.4}$$

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$z = x + jy = r\ \varphi$, $z_1 = x_1 + jy_1 = r_1 \angle\varphi 1$, $z_2 = x_2 + jy_2 = r_2 \angle\varphi 1$

The following operations are important.

Addition: $z_1 + z_2 = (x_1 + x_2) + j\,(y_1 + y_2)$ (7.5a)

Subtraction: $z_1 - z_2 = (x_1 - x_2) + j\,(y_1 - y_2)$ (7.5b)

Multiplication: $z_1 z_2 = r_1 r_2 \angle \varphi 1 + \varphi 2$ (7.5c)

Division: $\frac{z1}{z2} = \frac{r1}{r2} \angle \varphi 1 - \varphi 2$ (7.5d)

Reciprocal: $\frac{1}{z} = \frac{1}{r} \angle -\varphi 1$ (7.5e)

Square Root: $\sqrt{z} = \sqrt{r} \angle \varphi/2$ (7.5f)

Complex Conjugate: $z^* = x - jy = r \angle -\varphi = r\, e^{-j\varphi}$ (7.5g)

Note that from Eq. (7.5e),

$$1/j = -j \quad (7.5h)$$

The idea of phasor representation is based on Euler's identity. In general,

$$e^{\pm j\varphi} = \cos\varphi \pm j\sin\varphi \quad (7.6)$$

$$\cos\varphi = \mathrm{Re}(e^{j\varphi}) \quad (7.7a)$$

$$\sin\varphi = \mathrm{Im}(e^{j\varphi}) \quad (7.7b)$$

where Re and Im stand for the real part of and the imaginary part of.

Given a sinusoid $v(t) = V_m \cos(\omega t + \varphi)$, we use Eq. (7.7a) to express v(t) as

$$\mathbf{v(t)= V_m \cos(\omega t + \varphi) = Re(V_m e^{j(\omega t+\varphi)}) = Re(V_m e^{j\varphi} e^{j\omega t})} \quad (7.8)$$

Thus,

$$\mathbf{v(t) = Re(V e^{j\omega t})} \quad (7.9)$$

where

$$\mathbf{V = V_m e^{j\varphi} = V_m \angle\varphi} \quad (7.10)$$

V is thus the phasor representation of the sinusoid v(t), as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid.

One way of looking at Eqs. (7.9) and (7.10) is to consider the plot in Fig. 7.1(a) and (b) of the sinor $V e^{j\omega t} = V_m e^{j(\omega t+\varphi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction, as shown in. In other words, the entire complex plane is rotating at an angular velocity of ω. We may regard v(t) as the projection of the sinor Ve jωt on the real axis, as shown in Fig. 7.1(b). The value of the sinor at time t = 0 is the phasor V of the sinusoid v(t). The sinor may be regarded as a rotating phasor.

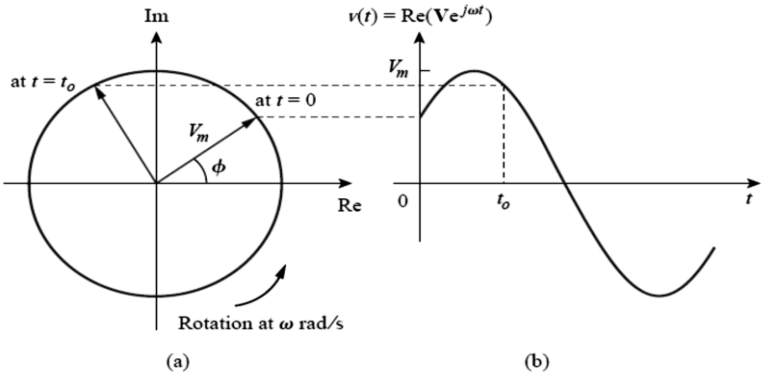

Figure 7.1 Representation of V$e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

For example, phasors $V = V_m \angle\varphi$ and $I = I_m \angle{-\theta}$ are graphically represented in Fig. 7.2. Such a graphical representation of phasors is known as a phasor diagram.

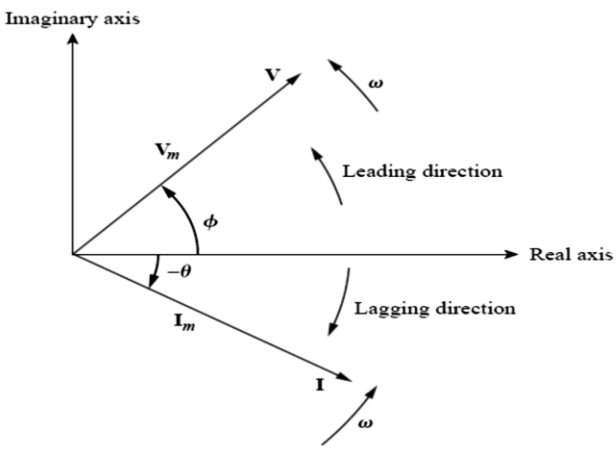

Figure 7.2 A Phasor diagram showing $V = V_m \angle\varphi$ and $I = I_m \angle{-\theta}$.

By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$\mathbf{v(t) = V_m \cos(\omega t + \varphi) \quad \Longleftrightarrow \quad V = V_m \angle\varphi} \tag{7.11}$$

(Time-domain representation) (Phasor-domain representation)

Note that in Eq. (7.11) the frequency (or time) factor e jωt is suppressed, and the frequency is not explicitly shown in the phasor-domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the frequency domain.

From Eqs. (7.9) and (7.10), $v(t) = \mathrm{Re}(V e^{j\omega t}) = V_m \cos(\omega t + \varphi)$, so that

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \varphi) = \omega V_m \cos(\omega t + \varphi + 90^\circ) = \mathrm{Re}(\omega V_m e^{j\omega t} e^{j\varphi} e^{j90^\circ}) = \mathrm{Re}(j\omega V e^{j\omega t}) \qquad (7.12)$$

This shows that the derivative v(t) is transformed to the phasor domain as $j\omega V$

$$\frac{dv}{dt}\ \text{(Timedomain)} \iff j\omega V\ \text{(Phasor domain)} \qquad (7.13)$$

Similarly, the integral of v(t) is transformed to the phasor domain as $V/j\omega$

$$\int v\,dt\ \text{(Timedomain)} \iff V/j\omega\ \text{(Phasor domain)} \qquad (7.14)$$

<u>Example 7.1:</u> Transform these sinusoids to phasors:

(a) $v = -4 \sin(30t + 50^\circ)$

(b) $i = 6 \cos(50t - 40^\circ)$

Solution: (a) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4 \sin(30t + 50^\circ) = 4 \cos(30t + 50^\circ + 90^\circ) = 4 \cos(30t + 140^\circ)$$

The phasor form of v is

$$V = 4 \angle 140^\circ$$

(b) $i = 6 \cos(50t - 40^\circ)$ has the phasor

$$I = 6 \angle -40^\circ$$

<u>Example 7.2:</u> Find the sinusoids represented by these phasors:

(a) $V = j8e^{-j20^\circ}$

(b) $I = -3 + j4$

<u>Solution:</u>

(a) Since $j = 1\angle 90^\circ$,

$$V = j8 \angle -20^\circ = (1 \angle 90^\circ)(8 \angle -20^\circ)$$

$$= 8 \angle (90^\circ - 20^\circ) = 8 \angle 70^\circ \text{ V}$$

Converting this to the time domain gives

$$v(t) = 8 \cos(\omega t + 70^\circ) \text{ V}$$

(b) $I = -3 + j4 = 5 \angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5 \cos(\omega t + 126.87^\circ) \text{ A}$$

7.2 Phasor Relationships for Circuit Elements

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention.

Resistor

We begin with the resistor. If the current through a resistor R is $i = I_m \cos(\omega t + \varphi)$, the voltage across it is given by Ohm's law as

$$\mathbf{v = iR = R\, I_m \cos(\omega t + \varphi)} \qquad (7.15)$$

The phasor form of this voltage is

$$\mathbf{V = R\, I_m \angle\varphi} \qquad (7.16)$$

But the phasor representation of the current is $I = Im \angle\varphi$. Hence,

$$\mathbf{V = RI} \qquad (7.17)$$

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 7.3 illustrates the voltage-current relations of a resistor. We should note from Eq. (7.17) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 7.4.

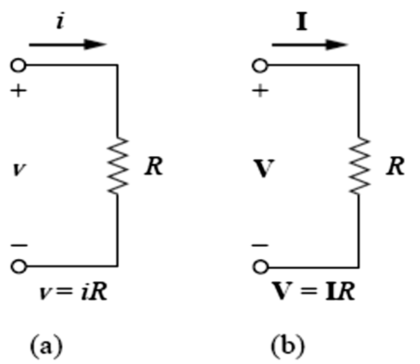

Figure 7.3 Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

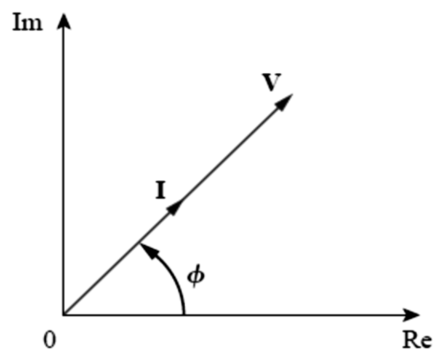

Figure 7.4 Phasor diagram for the resistor.

Inductor

For the inductor L, assume the current through it is $i = I_m \cos(\omega t + \varphi)$. The voltage across the inductor is

$$\mathbf{v = L\, di/dt = -\,\omega\, L\, I_m \sin(\omega t + \varphi)} \qquad (7.18)$$

Recall that $-\sin A = \cos(A + 90^\circ)$. We can write the voltage as

$$\mathbf{v = \omega\, L\, I_m \cos(\omega t + \varphi + 90^\circ)} \qquad (7.19)$$

which transforms to the phasor

$$\mathbf{V = \omega\, L\, I_m\, e^{j(\varphi+90^\circ)} = \omega\, L\, I_m\, e^{j\varphi}\, e^{j90^\circ} = \omega\, L I_m \angle\varphi\, e^{j90^\circ}} \qquad (7.20)$$

But $I_m\ \varphi = I$, and from Eq. (7.19), $e^{j90^\circ} = j$. Thus,

$$\mathbf{V = j\omega\, L\, I} \qquad (7.21)$$

showing that the voltage has a magnitude of ωLI m and a phase of $\varphi+90^\circ$. The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°. Figure 7.5 shows the voltage-current relations for the inductor. Figure 7.6 shows the phasor diagram.

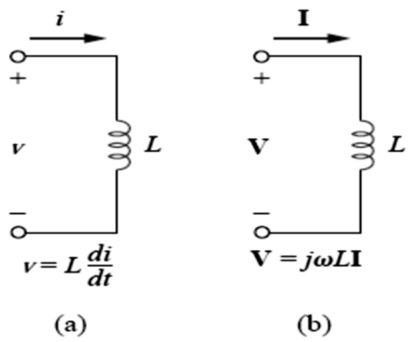

Figure 7.5 **Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.**

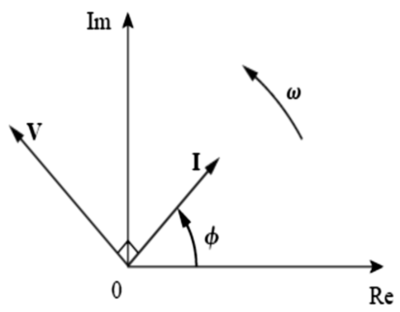

Figure 7.6 **Phasor diagram for the inductor; I lags V.**

Capacitor

For the capacitor C, assume the voltage across it is $v = V_m \cos(\omega t + \varphi)$. The current through the capacitor is

$$\mathbf{i = C\, dv/dt} \qquad (7.22)$$

By following the same steps as we took for the inductor or by applying Eq. (7.13) on Eq. (7.22), we obtain

$$\mathbf{I = j\omega C V \Rightarrow V = I / j\omega C} \qquad (7.23)$$

showing that the current and voltage are 90° out of phase. To be specific, the current leads the voltage by 90°. Figure 7.7 shows the voltage-current relations for the capacitor; Fig. 7.8 gives the phasor diagram.

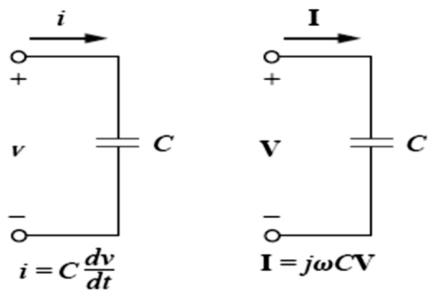

Figure 7.7 Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

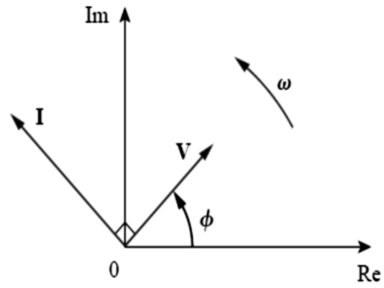

Figure 7.8 Phasor diagram for the capacitor; I leads V.

<u>Example 7.3:</u> The voltage v = 12 cos(60t + 45°) is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

<u>Solution:</u> For the inductor, V = jωLI, where ω = 60 rad/s and V = 12 ∠45° V. Hence

$$I = \frac{V}{j\omega L} = \frac{12\angle 45^\circ}{j60 \times 0.1} = \frac{12\angle 45^\circ}{6\angle 90^\circ} = 2\angle -45^\circ\,A$$

Converting this to the time domain,

i(t)= 2 cos(60t − 45°) A

7.3 Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

V = RI, V = jω L I, V = I /jω C (7.24)

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

V /I = R, V /I = jωL, V /I = 1/ jωC (7.25)

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

Z = V/I or V = ZI (7.26)

where Z is a frequency-dependent quantity known as impedance, measured in ohms.

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω).

The impedance represents the opposition which the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity. The impedances of resistors, inductors, and capacitors can be readily obtained from Eq. (7.25). we notice that $Z_L = j\omega L$ and $Z_C = -j/\omega C$. Consider two extreme cases of angular frequency. When $\omega = 0$ (i.e., for dc sources), $Z_L = 0$ and $Z_C \rightarrow \infty$, confirming what we already know— that the inductor acts like a short circuit, while the capacitor acts like an open circuit. When $\omega \rightarrow \infty$ (i.e., for high frequencies), $Z_L \rightarrow \infty$ and $Z_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit.

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z = R + jX} \tag{7.27}$$

where R = Re Z is the resistance and X = Im Z is the reactance. The reactance X may be positive or negative. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$\mathbf{Z = |Z| \angle\theta} \tag{7.28}$$

Comparing Eqs. (7.27) and (7.28), we infer that

$$\mathbf{Z = R + jX = |Z| \angle\theta} \tag{7.29}$$

Where $$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R} \tag{7.30}$$

and

$$\mathbf{R = |Z| \cos\theta, \quad X = |Z| \sin\theta} \tag{7.31}$$

It is sometimes convenient to work with the reciprocal of impedance, known as admittance.

The admittance Y is the reciprocal of impedance, measured in siemens (S).

The admittance Y of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$\mathbf{Y = 1/Z = I/V} \tag{7.32}$$

The admittances of resistors, inductors, and capacitors can be obtained from Eq. (7.29). As a complex quantity, we may write Y as

$$\mathbf{Y = G + jB} \tag{7.33}$$

where G = Re Y is called the conductance and B = Im Y is called the susceptance. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (7.27) and (7.33),

$$\mathbf{G + jB} = \frac{1}{R + jX} \qquad (7.34)$$

the real and imaginary parts gives

$$\mathbf{G} = \frac{R}{R^2 + X^2}, \mathbf{B} = \frac{-X}{R^2 + X^2} \qquad (7.35)$$

showing that G ≠ 1/R as it is in resistive circuits. Of course, if X = 0, then G = 1/R.

Example 7.4: Find v(t) and i(t) in the circuit shown in Fig. 7.9.

Solution: From the voltage source 10 cos 4t , ω = 4, Vs = 10 ∠0 ◦ V The impedance is

$$Z = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$I = \frac{Vs}{Z} = \frac{10\ \angle 0°}{5 - j\,2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$

$$= 1.6 + j0.8 = 1.789\ \angle 26.57°\ \text{A} \qquad (7.4.1)$$

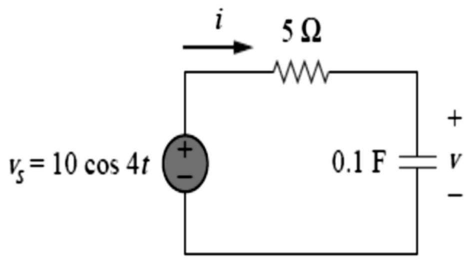

Fig. 7.9

The voltage across the capacitor is

$$V = I\,ZC = \frac{I}{j\omega C} = \frac{1.789\ \angle 26.57°}{j4 \times 0.1}$$

$$= \frac{1.789\ \angle 26.57°}{0.4\angle \mathbf{90}°} = 4.47\ \angle -63.43°\ \text{V} \qquad (7.4.2)$$

Converting I and V in Eqs. (7.4.1) and (7.4.2) to the time domain, we get

i(t) = 1.789 cos(4t + 26.57 ◦) A

v(t) = 4.47 cos(4t − 63.43 ◦) V

Notice that i(t) leads v(t) by 90 ◦ as expected.

7.4 Kirchhoff's Laws in The Frequency Domain

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. Therefore, we need to express them in the frequency domain.

For KVL, let $\mathbf{v_1, v_2,\ldots,v_n}$ be the voltages around a closed loop. Then

$$\mathbf{v_1 + v_2 + \cdots + v_n = 0} \qquad (7.36)$$

In the sinusoidal steady state, each voltage may be written in cosine form, so that Eq. (7.36) becomes

$$\mathbf{V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0} \qquad (7.37)$$

This can be written as

$$\mathbf{Re(V_{m1}\, e^{j\theta 1}\, e^{j\omega t}) + Re(V_{m2}\, e^{j\theta 2}\, e^{j\omega t}) + \cdots + Re(V_{mn}\, e^{j\theta n}\, e^{j\omega t}) = 0}$$

or

$$\mathbf{Re[(V_{m1}\, e^{j\theta 1} + V_{m2}\, e^{j\theta 2} + \cdots + V_{mn}\, e^{j\theta n})\, e^{j\omega t}] = 0} \qquad (7.37)$$

If we let $V_k = V_{mk}\, e^{j\theta k}$, then

$$\mathbf{Re[(V_1 + V_2 + \cdots + V_n)\, e^{j\omega t}] = 0} \qquad (7.38)$$

Since e jωt $\neq$ 0,

$$\mathbf{V_1 + V_2 + \cdots + V_n = 0} \qquad (7.39)$$

indicating that Kirchhoff's voltage law holds for phasors.

By following a similar procedure, we can show that Kirchhoff's current law holds for phasors. If we let $i_1, i_2,\ldots,i_n$ be the current leaving or entering a closed surface in a network at time t , then

$$\mathbf{i_1 + i_2 + \cdots + i_n = 0} \qquad (7.40)$$

If $I_1, I_2,\ldots, I_n$ are the phasor forms of the sinusoids i_1 , i_2 ,..., i_n , then

$$\mathbf{I_1 + I_2 + \cdots + I_n = 0} \qquad (7.41)$$

which is Kirchhoff's current law in the frequency domain. Once we have shown that both KVL and KCL hold in the frequency domain, it is easy to do many things, such as impedance combination, nodal and mesh analyses, superposition, and source transformation.

7.5 Impedance Combinations

Consider the N series-connected impedances shown in Fig. 7.10. The same current I flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V = V_1 + V_2 + \cdots + V_N = I(Z_1 + Z_2 + \cdots + Z_N)} \qquad (7.42)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z_{eq} = Z_1 + Z_2 + \cdots + Z_N} \qquad (7.43)$$

showing that the total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.

We can use voltage-division relationship to calculate voltage across each impedance, then

$$V_N = \frac{V Z_N}{Z_1 + Z_2 + \cdots + Z_N} \qquad (7.44)$$

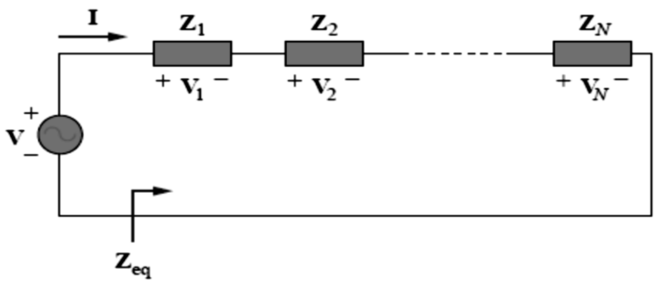

Figure 7.10 *N* impedances in series.

In the same manner, we can obtain the equivalent impedance or admittance of the N parallel-connected impedances shown in Fig. 7.11. The voltage across each impedance is the same. Applying KCL at the top node,

$$I = I_1 + I_2 + \cdots + I_N = V\left(\frac{1}{Z1} + \frac{1}{Z2} + \cdots + \frac{1}{ZN}\right) \qquad (7.45)$$

The equivalent impedance is

$$\frac{1}{Zeq} = \frac{I}{V} = \frac{1}{Z1} + \frac{1}{Z2} + \cdots + \frac{1}{ZN} \qquad (7.46)$$

and the equivalent admittance is

$$\mathbf{Yeq = Y1 + Y2 + \cdots + YN} \qquad (7.47)$$

This indicates that the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances.

When N = 2, as shown in Fig. 7.12, the currents in the impedances are

$$I1 = \frac{Z2}{Z1 + Z2} I, \; I2 = \frac{Z1}{Z1 + Z2} I \qquad (7.48)$$

which is the current-division principle.

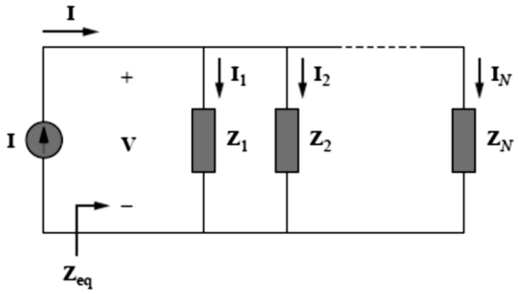

Figure 7.11 N impedances in parallel.

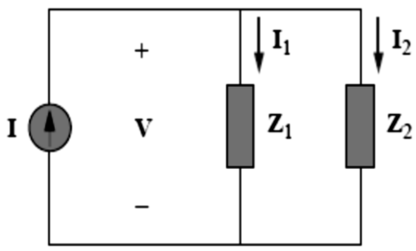

Figure 7.12 Current division.

The delta-to-wye and wye-to-delta transformations that we applied to resistive circuits are also valid for impedances. With reference to Fig. 7.13, the conversion formulas are as follows.

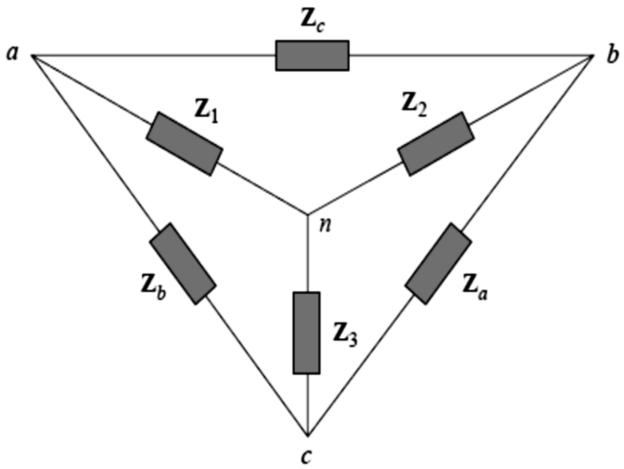

Figure 7.13 Superimposed Y and Δ networks.

Y -Δ Conversion:

$$Za = \frac{Z1Z2 + Z2Z3 + Z3Z1}{Z1}$$

$$Zb = \frac{Z1\,Z2 + Z2\,Z3 + Z3\,Z1}{Z2}$$

$$Zc = \frac{Z1\,Z2 + Z2\,Z3 + Z3\,Z1}{Z3} \qquad (7.49)$$

Δ-Y Conversion:

$$Z1 = \frac{Zb\,Zc}{Za + Zb + Zc}$$

$$Z2 = \frac{Zc\,Za}{Za + Zb + Zc}$$

$$Z3 = \frac{ZaZb}{Za + Zb + Zc} \qquad (7.50)$$

A delta or wye circuit is said to be balanced if it has equal impedances in all three branches.

When a Δ-Y circuit is balanced, Eqs. (7.49) and (7.50) become

$$\mathbf{Z\Delta = 3ZY \quad or \quad ZY = \frac{1}{3}\, Z\Delta} \tag{7.51}$$

where ZY = Z1 = Z2 = Z3 and ZΔ = Za = Zb = Zc.

Example 7.5: Find the input impedance of the circuit in Fig. 7.14. Assume that the circuit operates at ω = 50 rad/s.

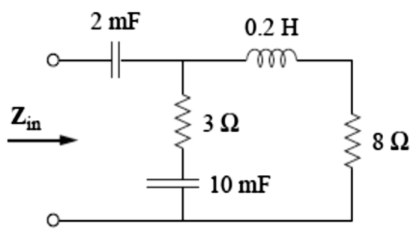

Figure 7.14 For Example 7.5.

Solution:

Let

Z1 = Impedance of the 2-mF capacitor

Z2 = Impedance of the 3-Ω resistor in series with the 10-mF capacitor

Z3 = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$Z1 = \frac{1}{j\omega C} = \frac{1}{j\,50 \times 2 \times 10^{-3}} = -j10\,\Omega$$

$$Z2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j\,50 \times 10 \times 10^{-3}} = (3 - j2)\,\Omega$$

$$Z3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j\,10)\,\Omega$$

The input impedance is

$$Zin = Z1 + Z2 \,||\, Z3 = -j\,10 + \frac{(3 - j2)(8 + j10)}{11 + j\,8}$$

$$= -j\,10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j\,10 + 3.22 - j1.07\,\Omega$$

Thus,

$$Zin = 3.22 - j11.07\,\Omega$$

Example 7.6: Determine vo(t) in the circuit in Fig. 7.15.

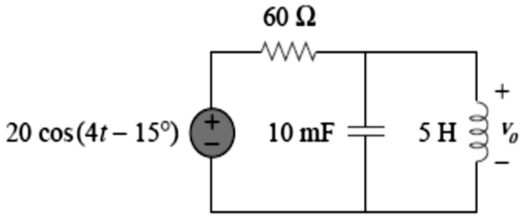

Figure 7.15 For Example 7.6.

Solution: To do the analysis in the frequency domain, we must first transform the time domain circuit in Fig. 7.15 to the phasor-domain equivalent in Fig. 7.16. The transformation produces

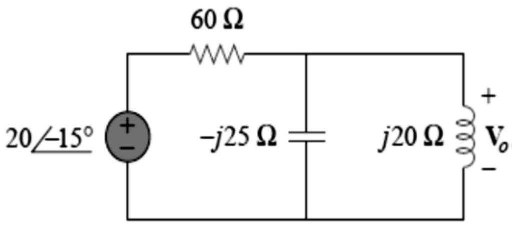

Figure 7.16 The frequency-domain equivalent of the circuit in Fig. 7.15.

$$vs = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad Vs = 20\angle{-15^\circ}\ \text{V},\ \omega = 4$$

$$10\ \text{mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j\,25\ \Omega$$

$$5\text{H} \quad \Rightarrow \quad j\omega L = j\,4 \times 5 = j\,20\ \Omega$$

Let

Z1 = Impedance of the 60-Ω resistor

Z2 = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor Then Z 1 = 60 Ω and

$$Z2 = -j\,25 \,||\, j\,20 = \frac{-j25 \times j\,20}{-j25 + j\,20} = j\,100\ \Omega$$

By the voltage-division principle,

$$Vo = \frac{Z2}{Z1 + Z2}\,Vs = \frac{j100}{60 + j\,100}\,(20\ \ -15^\circ)$$

$$= (0.8575\ \angle 30.96^\circ)\,(20\ \angle{-15^\circ}) = 17.15\ \angle 15.96^\circ\ \text{V}.$$

We convert this to the time domain and obtain

$$vo\,(t) = 17.15\cos(4t + 15.96^\circ)\text{V}$$

Example 7.7: Find current I in the circuit in Fig. 7.17.

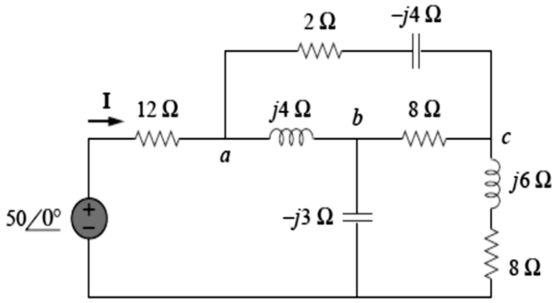

Figure 7.17 For Example 7.7.

Solution:

The Δ network connected to nodes a, b, and c can be converted to the Y network of Fig. 7.17. We obtain the Y impedances as follows using Eq. (7.50):

$$\mathbf{Zan} = \frac{j4(2-j4)}{j4+2-j4+8} = \frac{4(4+j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Zbn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Zcn} = \frac{8(2-j4)}{10} = (1.6 - j\,3.2)\ \Omega$$

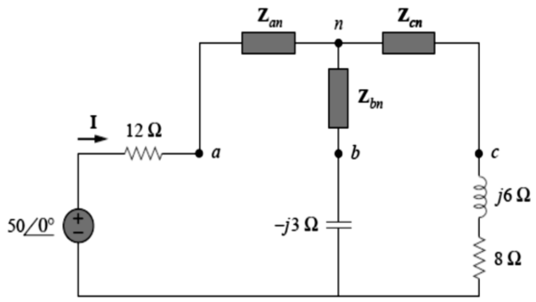

Figure 7.18 The circuit in Fig. 7.17 after Δ-to-Y transformation.

The total impedance at the source terminals is

$$\mathbf{Z} = 12 + \mathbf{Zan} + (\mathbf{Zbn} - j3) \parallel (\mathbf{Zcn} + j6 + 8)$$

$$= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8)$$

$$= 13.6 + j0.8 + \frac{j\,0.2(9.6+j2.8)}{9.6+j3}$$

$$= 13.6 + j1 = 13.64\ \angle 4.204^\circ$$

The desired current is

$$\mathbf{I} = \frac{V}{Z} = \frac{50\ \angle 0^\circ}{13.64\ \angle 4.204^\circ} = 3.666\ \angle -4.204^\circ\ \mathrm{A}$$