اسم المادة : رياضيات-2
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عنوان المحاضرة:

المشتقات القيمية والاتجاهية Gradients and Directional Derivatives

Gradients and Directional Derivatives :-

(1) If the partial derivatives of $f(x,y,z)$ are defined at $P_0(x_0,y_0,z_0)$; then the gradient of f at P_0 is the vector

$$\boxed{\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k}$$

(2) The directional derivative is defined as

$$\boxed{(D_u f)_{P_0} = (\nabla f)_{P_0} \cdot u}$$

where u is a unit vector

Ex:- Find the directional derivative of $f(x,y,z) = xy^2z^3$ at the point $P(3,2,1)$ in the direction towards $Q(5,3,2)$.

Solution:-

$$\overrightarrow{PQ} = (5-3)i + (3-2)j + (2-1)k = \boxed{2i + j + k}$$

$$u = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{2i+j+k}{\sqrt{2^2+1^2+1^2}} = \boxed{\frac{1}{\sqrt{6}}(2i+j+k)}$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= y^2z^3 i + 2xyz^3 j + 3xy^2z^2 k$$

$$(\nabla f)_p = (2^2 * 1^3) i + (2*3*2*1^3) j + (3*3*2^2*1^2) k$$

$$= \boxed{4i + 12j + 36k}$$

$$\therefore (D_u f)_p = (\nabla f)_p \cdot u$$

$$= (4i + 12j + 36k) \cdot \left(\frac{1}{\sqrt{6}}(2i + j + k)\right)$$

$$= \frac{1}{\sqrt{6}}(8 + 12 + 36) = \boxed{\frac{56}{\sqrt{6}}}$$

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Ex:- Find the directional derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_o(1,1,0)$ in the direction of vector $A = 2i - 3j + 6k$.

Solution:-

$$U = \frac{\vec{A}}{|A|} = \frac{2i - 3j + 6k}{\sqrt{(2)^2 + (-3)^2 + (6)^2}} = \frac{2i - 3j + 6k}{7}$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= (3x^2 - y^2) i + (-2xy) j + (-1) k$$

$$(\nabla f)_{P_o} = (3*1^2 - 1^2) i + (-2*1*1) j - k$$

$$= 2i - 2j - k$$

$$\therefore (D_u f)_{P_o} = (\nabla f)_{P_o} \cdot u$$

$$= (2i - 2j - k) \cdot \left(\frac{2i - 3j + 6k}{7}\right)$$

$$= \frac{4 + 6 - 6}{7} = \boxed{\frac{4}{7}}$$

Note:-

(1) If $f(x,y,z)$ = function then

$$\overrightarrow{\text{grad}}\, f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

(2) If $\vec{F}(x,y,z) = f_1(x,y,z) i + f_2(x,y,z) j + f_3(x,y,z) k$ then

$$\text{Divergence}\, \vec{F} = \vec{\nabla} \cdot \vec{F} = \text{div}\, \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\therefore \overrightarrow{\text{Curl}}\, F = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

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Maxima, Minima and Saddle Points :-

If $f(x,y)$ is a function of two independent variables (x,y) and the interior points (a,b) are found at $f_x = f_y = 0$. then:

1. f has a local maximum at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)

2. f has a local minimum at (a,b) if $f_{xx} > 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ at (a,b)

3. f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b) where (a,b) is the critical point

Ex:- Find the extreme value of $f(x,y) = x^2 + y^2$

Solution:-

$f_x = 2x \quad , \quad f_x = 0 = 2x \Rightarrow x = 0 = a$

$f_y = 2y \quad , \quad f_y = 0 = 2y \Rightarrow y = 0 = b$

$f_{xx} = 2 \quad , \quad f_{yy} = 2 \quad , \quad f_{xy} = 0 \Rightarrow f_{xy}^2 = 0$

$f_{xx} f_{yy} - f_{xy}^2 = 2 * 2 - 0 = 4 > 0$

$\because f_{xx} > 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$

and $(a,b) = (0,0)$ is the critical point

$\therefore f(x,y) = x^2 + y^2$ has a local minimum at (a,b)

$f(0,0) = 0$

Ex:- Find the extreme Value of the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Solution:-

$f_x = y - 2x - 2 \Rightarrow \boxed{f_{xx} = -2}$

$f_x = 0 \Rightarrow y - 2x - 2 = 0$ —— (1)

$f_y = x - 2y - 2 \Rightarrow \boxed{f_{yy} = -2}$

$f_y = 0 \Rightarrow x - 2y - 2 = 0$ —— (2)

لايجاد قيمة X و y من المعادلتين (1 و 2) يتم حل المعادلتين آنياً

$y - 2x - 2 = 0$ —— (1)
$-2y + x - 2 = 0$ —— (2) $*2$

$y - 2x - 2 = 0$ —— (1)
$-4y + 2x - 4 = 0$ —— (2)

بالجمع

$-3y - 6 = 0 \Rightarrow 3y = -6 \Rightarrow y = -2 = b$

نعوض قيمة y في المعادلة (1) لايجاد قيمة X

$-2 - 2x - 2 = 0 \Rightarrow -4 = 2x$

$x = -2 = a$

$\therefore (a,b) = (-2,-2)$

$f_{xx} f_{yy} - f_{xy}^2$

$-2 * -2 - (1)^2 = 3 > 0$

$\boxed{\begin{array}{l} * f_{xx} = -2 \\ * f_{yy} = -2 \end{array}}$

$\because f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

$\therefore f(x,y)$ has a local maximum at $(-2,-2)$

$f(-2,-2) = 8$

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Lagrange Multipliers :-

The extreme values of a function $f(x,y,z)$ whose variable are subject to a constraint of form $g(x,y,z)=0$ are to be found on the surface $g=0$ at the points where

$$\nabla f = \lambda \nabla g$$

λ called a Lagrange multiplier.

Ex:- Find the greatest and smallest values that the function $f(x,y)=xy$ takes on the ellipes

$$\frac{x^2}{8}+\frac{y^2}{2}=1$$

Solution:-

$$g(x)=\frac{x^2}{8}+\frac{y^2}{2}-1=0$$

$$\nabla f=\lambda\nabla g$$

$$\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j=\lambda\left(\frac{\partial g}{\partial x}i+\frac{\partial g}{\partial y}j\right)$$

$$yi+xj=\lambda\left(\frac{2x}{8}i+\frac{2y}{2}j\right)$$

$$y=\lambda\frac{x}{4} \quad \text{—(1)}$$

$$x=\lambda y \quad \text{—(2)} \quad \text{sub in eq (1)}$$

$$y=\lambda * \frac{\lambda y}{4}=\frac{\lambda^2 y}{4}$$

$$\frac{\lambda^2}{4}=1 \Rightarrow \lambda^2=4 \Rightarrow \boxed{\lambda=\mp 2}$$

or $y=0$

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Case I :- If $y=0$ then $x=y=0$

The point $(0,0)$ is not on the ellipes

Case II :- If $y\neq 0$ then $\lambda=\mp 2$

$$x=\mp 2y$$

Sub. this $(x=\mp 2y)$ in the eq $g(x,y)=0$ gives

$$\frac{(\mp 2y)^2}{8}+\frac{y^2}{2}=1$$

$$\frac{4y^2+4y^2}{8}=1 \Rightarrow 8y^2=8$$

$$y^2=1 \Rightarrow y=\mp 1$$

$$f(x,y)=f(\mp 2,\mp 1)$$

$$f(\mp 2,1)=\mp 2$$

$$f(\mp 2,-1)=\mp 2$$

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H.W

1) Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if $f(x,y) = x^2 + 3xy + y - 1$

2) Find $\partial f/\partial x$ and $\partial f/\partial y$ for $f(x,y) = (x^2-1)(y+2)$

3) Find $\partial f/\partial x$ and $\partial f/\partial y$ for $f(x,y) = \frac{x}{x^2+y^2}$

4) Find f_x, f_y and f_z for

a/ $f(x,y,z) = \ln(x+2y+3z)$

b/ $f(x,y,z) = e^{-(x^2+y^2+z^2)}$

5) Find all the second order partial derivatives of the $f(x,y) = \sin(xy)$

6) Find the value of df/dt at $t=0$ if $f(x,y) = x^2+y^2$ and $x = \cos t + \sin t$, $y = \cos t - \sin t$.

7) Find the value of df/dt at $t=3$ if $f(x,y,z) = \ln(x^2+y^2+z^2)$ and $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$

8) Find ∇f at the point $(1,1,1)$ if $f(x,y,z) = x^2+y^2-2z^2+z\ln x$

9) Find the derivative of $f(x,y,z) = xy+yz+zx$ at $P_0(1,-1,2)$ in the direction of $\vec{v} = 3i+6j-2k$.

10) Find the local minima, local maxima and Saddle points of $f(x,y) = x^2+2xy$

11) Find the maximum values of $f(x,y) = 49-x^2-y^2$ on the line $x+3y=10$

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