عنوان المحاضرة:

التكاملات الثنائية، التطبيقات الفيزياوية Double Intergrals, Physical Applications

Areas, Moments and Center of Mass :—

The area of a closed bounded plane region R is:

$$A = \iint_R dA = \iint_R dx\, dy = \iint_R dy\, dx$$

To find the moments and Center of mass of thin sheet or plate we use the formulas :—

① Mass :—

$$M = \iint \delta(x,y)\cdot dA$$

where $\delta(x,y)$ is the density

② First moments :—

$$M_x = \iint y\cdot \delta(x,y)\cdot dA$$

$$M_y = \iint x\cdot \delta(x,y)\cdot dA$$

③ Center of mass :—

$$\bar{X} = M_y / M$$

$$\bar{y} = M_x / M$$

④ Moment of inertia :—

- About x-axis $\quad I_x = \iint y^2\cdot \delta(x,y)\cdot dA$

- About y-axis $\quad I_y = \iint x^2\cdot \delta(x,y)\cdot dA$

- About the origin $\quad I_0 = \iint (x^2+y^2)\cdot \delta(x,y)\cdot dA$

⑤ Radii of gyration :—

- About x-axis $\quad R_x = \sqrt{I_x/M}$

- About y-axis $\quad R_y = \sqrt{I_y/M}$

- About the origin $\quad R_0 = \sqrt{I_0/M}$

6

EX:— A thin plate covers the plane region R bounded by $y=x^2$, $y=\sqrt{2-x^2}$ and y-axis and with a density function
$$\delta = xy$$

a- Find the mass of the body
b- Find the center of the mass
c- Find the radii of gyration

Solution:—

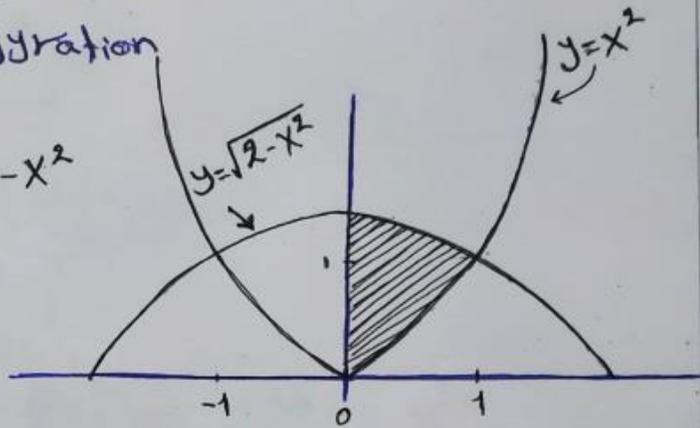$$x^2 = \sqrt{2-x^2} \Rightarrow x^4 = 2-x^2$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0$$

$$x^2 - 1 = 0 \Rightarrow x^2 = 1$$

$$\therefore \boxed{x = \mp 1}$$

$$x^2 + 2 = 0 \Rightarrow x^2 = -2 \quad \text{مهمل}$$

$$\therefore \boxed{y = 1^2 = 1}$$

a/

$$M = \iint_R \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} xy \cdot dy\, dx$$

$$= \int_0^1 x \frac{y^2}{2}\Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 x\left[\left(\frac{(\sqrt{2-x^2})^2}{2}\right) - \frac{(x^2)^2}{2}\right] dx$$

$$= \int_0^1 x \cdot \left[\frac{2-x^2}{2} - \frac{x^4}{2}\right] dx$$

$$= \int_0^1 \left[x - \frac{x^3}{2} - \frac{x^5}{2}\right] dx = \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{12}\Big]_0^1$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12} = \boxed{\frac{7}{24}}$$

$\boxed{7}$

$$b/ \; M_x = \iint_R y \cdot \delta(x,y) \cdot dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x y^2 \cdot dy \, dx$$

$$= \int_0^1 x \frac{y^3}{3} \Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx$$

$$= \int_0^1 x \cdot \left[\left(\frac{(\sqrt{2-x^2})^3}{3} \right) - \frac{(x^2)^3}{3} \right] \cdot dx$$

$$= \int_0^1 \left[\frac{x (2-x^2)^{3/2}}{3} - \frac{x^7}{3} \right] \cdot dx$$

$$= \frac{-1}{2*3} \cdot \frac{(2-x^2)^{5/2}}{5/2} - \frac{x^8}{8*3} \Big]_0^1 =$$

$$= \left[\left(\frac{-1}{15} (2-1)^{5/2} - \frac{1^8}{24} \right) \right] - \left[\left(\frac{-1}{15} (2-0^2)^{5/2} \right) - \frac{0^8}{24} \right]$$

$$= \boxed{0.268}$$

$$\therefore \; \bar{y} = M_x / M \; = \frac{0.268}{7/24} = \boxed{0.922}$$

$$M_y = \iint_R x \cdot \delta(x,y) \, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x^2 y \, dy \cdot dx$$

$$= \int_0^1 x^2 \frac{y^2}{2} \Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 x^2 \left[\frac{(\sqrt{2-x^2})^2}{2} - \frac{(x^2)^2}{2} \right] \cdot dx$$

$$= \int_0^1 x^2 \left[\frac{2-x^2}{2} - \frac{x^4}{2} \right] \cdot dx = \int_0^1 \left(x^2 - \frac{x^4}{2} - \frac{x^6}{2} \right) \cdot dx$$

$$= \left(\frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{14} \right) \Big]_0^1 = \frac{1}{3} - \frac{1}{10} - \frac{1}{14} = \boxed{\frac{17}{105}}$$

$$\therefore \; \bar{X} = M_y / M \; = \frac{17/105}{7/24} = \boxed{0.56}$$

$$\boxed{8}$$

C/ $I_x = \iint\limits_R y^2 \cdot \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x y^3 \cdot dy \cdot dx$

$= \int_0^1 x \frac{y^4}{4} \Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 \frac{x}{4} \left[(\sqrt{2-x^2})^4 - (x^2)^4 \right] \cdot dx$

$= \int_0^1 \left[\frac{x}{4}(2-x^2)^2 - \frac{x^9}{4} \right] dx = \frac{-1}{4*2} \frac{(2-x^2)^3}{3} - \frac{x^{10}}{40} \Big]_0^1$

$= \left[\frac{-1}{24}(2-1^2)^3 - \frac{1}{40} \right] - \left[\frac{-1}{24}(2-0)^3 - \frac{0}{40} \right]$

$= \frac{-1}{24} - \frac{1}{40} + \frac{8}{24} = \frac{4}{15} = \boxed{0.27}$

$\therefore R_x = \sqrt{I_x / M} = \sqrt{\frac{4/15}{7/24}} = \boxed{0.96}$

$I_y = \iint\limits_R x^2 \cdot \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x^3 y \cdot dy \cdot dx$

$= \int_0^1 x^3 \frac{y^2}{2} \Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 \frac{x^3}{2} \left[(\sqrt{2-x^2})^2 - (x^2)^2 \right] \cdot dx$

$= \int_0^1 \left[x^3 - \frac{x^5}{2} - \frac{x^7}{2} \right] dx = \frac{x^4}{4} - \frac{x^6}{12} - \frac{x^8}{16} \Big]_0^1$

$= \frac{1}{4} - \frac{1}{12} - \frac{1}{16} = \boxed{\frac{5}{48}}$

$\therefore R_y = \sqrt{I_y / M} = \sqrt{\frac{5/48}{7/24}} = \boxed{0.597}$

$I_o = I_x + I_y = \frac{4}{15} + \frac{5}{48} = \boxed{\frac{89}{240}}$

$R_o = \sqrt{I_o / M} = \sqrt{\frac{89/240}{7/24}} = \boxed{1.13}$

$\boxed{9}$

----- نهاية محاضرة "**التكاملات الثنائية، التطبيقات الفيزياوية** **Double Intergrals, Physical Applications**" -----

E-mail: hussein.kadhim@uomus.edu.iq