Natural Frequency of Free Transverse Vibrations For a Shaft Subjected to a Number of Point Loads

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INTRUODACTION :

Consider a shaft AB of negligible mass loaded with point loads W1 , W2, W3 and W4 etc. in newtons, as shown in Fig.1. Let m1, m2, m3 and m4 etc. be the corresponding masses in kg.

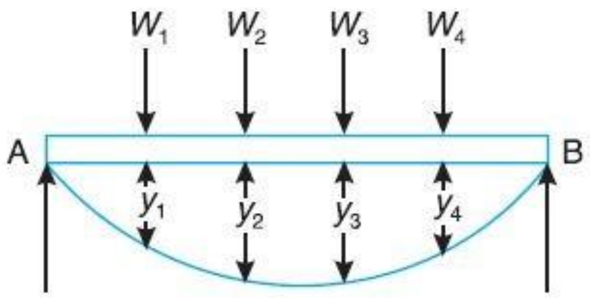

Fig. 1. Shaft carrying a number of point loads.

The natural frequency of such a shaft may be found out by the following two methods: طرق ايجاد التردد الطبيعي للاهتزاز العرضي لاحمال النقطة

1. Energy (or Rayleigh's) method طريقة الطاقة (طريقة رالي)

Let y1, y2, y3, y4 etc. be total deflection under loads W1, W2, W3 and W4 etc. as shown in Fig.1. We know that maximum potential energy اكبر طاقه كامنه

$$= \frac{1}{2} \times m_1 . g . y_1 + \frac{1}{2} \times m_2 . g . y_2 + \frac{1}{2} m_3 . g . y_3 + \frac{1}{2} \times m_4 . g . y_4 + ...$$

$$= \frac{1}{2} \Sigma m . g . y$$

اكبر طاقة حركية

and maximum kinetic energy

$$= \frac{1}{2} \times m_1 (\omega . y_1)^2 + \frac{1}{2} \times m_2 (\omega . y_2)^2 + \frac{1}{2} \times m_3 (\omega . y_3)^2 + \frac{1}{2} \times m_4 (\omega . y_4)^2 + ...$$

$$= \frac{1}{2} \times \omega^2 \left[m_1 (y_1)^2 + m_2 (y_2)^2 + m_3 (y_3)^2 + m_4 (y_4)^2 + ... \right]$$

$$= \frac{1}{2} \times \omega^2 \Sigma m . y^2 \qquad \text{. . . (where } \omega = \text{Circular frequency of vibration)}$$

Equating the maximum kinetic energy to the maximum potential energy, we have

$$\frac{1}{2} \times \omega^2 \, \Sigma m . y^2 = \frac{1}{2} \Sigma m . g . y$$

$$\therefore \quad \omega^2 = \frac{\Sigma m . g . y}{\Sigma m . y^2} = \frac{g \, \Sigma m . y}{\Sigma m . y^2} \qquad \text{or} \qquad \omega = \sqrt{\frac{g \, \Sigma m . y}{\Sigma m . y^2}}$$

$\therefore$ Natural frequency of transverse vibration,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g \, \Sigma m . y}{\Sigma m . y^2}}$$

2. Dunkerley’s method طريقة دنكرلي

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The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley’s empirical formula. According to this

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + \dots + \frac{1}{(f_{ns})^2}$$

where f_n = Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load.

f_{n1}, f_{n2}, f_{n3}, etc. = Natural frequency of transverse vibration of each point load.

f_{ns} = Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft).

Now, consider a shaft *AB* loaded as shown in Fig. 2.

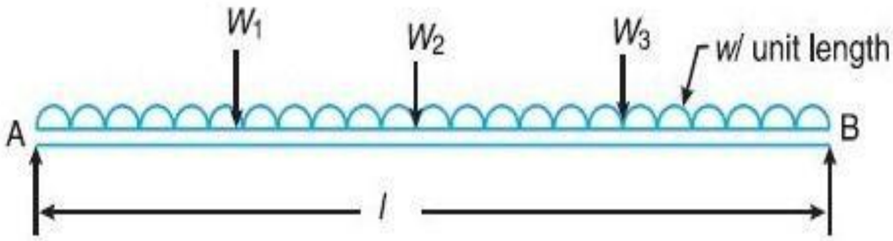


Fig. 2 Shaft carrying a number of point loads and a uniformly distributed load.

Let δ1,δ2,δ3, etc. = Static deflection due to the load W1, W2, W3 etc. when considered separately

to the δS = Static deflection due to the uniformly distributed load or due mass of the shaft .

We know that natural frequency of transverse vibration due to load W1,

$$f_{n_1} = \frac{0.4985}{\sqrt{\delta_1}} \text{ Hz}$$

Similarly, natural frequency of transverse vibration due to load W_2,

$$f_{n_2} = \frac{0.4985}{\sqrt{\delta_2}} \text{ Hz}$$

and, natural frequency of transverse vibration due to load W_3,

$$f_{n_3} = \frac{0.4985}{\sqrt{\delta_3}} \text{ Hz}$$

Also natural frequency of transverse vibration due to uniformly distributed load or weight of the shaft,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz}$$

Therefore, according to Dunkerley's empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^3} + \dots + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \dots + \frac{\delta_S}{(0.5615)^2}$$

$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \dots + \frac{\delta_S}{1.27}\right]$$

or
$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \ldots + \frac{\delta_S}{1.27}}} \text{ Hz}$$

Notes : 1. When there is no uniformly distributed load or mass of the shaft is negligible, then $\delta_S = 0$.

$$\therefore \quad f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \ldots}} \text{ Hz}$$

2. The value of δ_1, δ_2, δ_3 etc. for a simply supported shaft may be obtained from the relation

$$\delta = \frac{Wa^2b^2}{3\,EIl}$$

where δ = Static deflection due to load W,
a and b = Distances of the load from the ends,
E = Young's modulus for the material of the shaft,
I = Moment of inertia of the shaft, and
l = Total length of the shaft.

Example.1.

A shaft 50 mm diameter and 3 meters long is simply supported at the ends and carries three loads of 1000 N, 1500 N and 750 N at 1 m, 2 m and 2.5 m from the left support. The Young's modulus for shaft material is 200 GN/m2. Find the frequency of transverse vibration.

Solution. Given : d = 50 mm = 0.05 m ; l = 3 m, W_1 = 1000 N ; W_2 = 1500 N ; W_3 = 750 N; E = 200 GN/m^2 = 200 × 10^9 N/m^2

The shaft carrying the loads is shown in Fig.1 We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

and the static deflection due to a point load W,

$$\delta = \frac{Wa^2b^2}{3\,EIl}$$

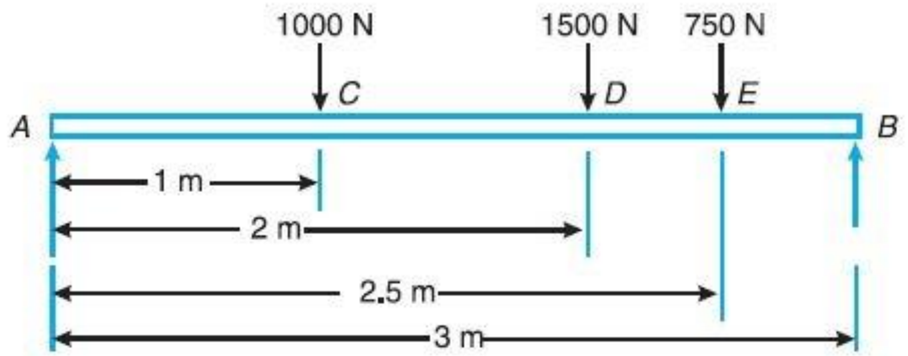


Fig .1

∴ Static deflection due to a load of 1000 N,

$$\delta_1 = \frac{1000\times1^2\times2^2}{3\times200\times10^9\times0.307\times10^{-6}\times3} = 7.24\times10^{-3} \text{ m}$$

. . . (Here $a = 1$ m, and $b = 2$ m)

Similarly, static deflection due to a load of 1500 N,

$$\delta_2 = \frac{1500\times2^2\times1^2}{3\times200\times10^9\times0.307\times10^{-6}\times3} = 10.86\times10^{-3} \text{ m}$$

. . . (Here $a = 2$ m, and $b = 1$ m)

and static deflection due to a load of 750 N,

$$\delta_3 = \frac{750(2.5)^2(0.5)^2}{3\times200\times10^9\times0.307\times10^{-6}\times3} = 2.12\times10^{-3} \text{ m}$$

. . . (Here $a = 2.5$ m, and $b = 0.5$ m)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1+\delta_2+\delta_3}} = \frac{0.4985}{\sqrt{7.24\times10^{-3}+10.86\times10^{-3}+2.12\times10^{-3}}}$$

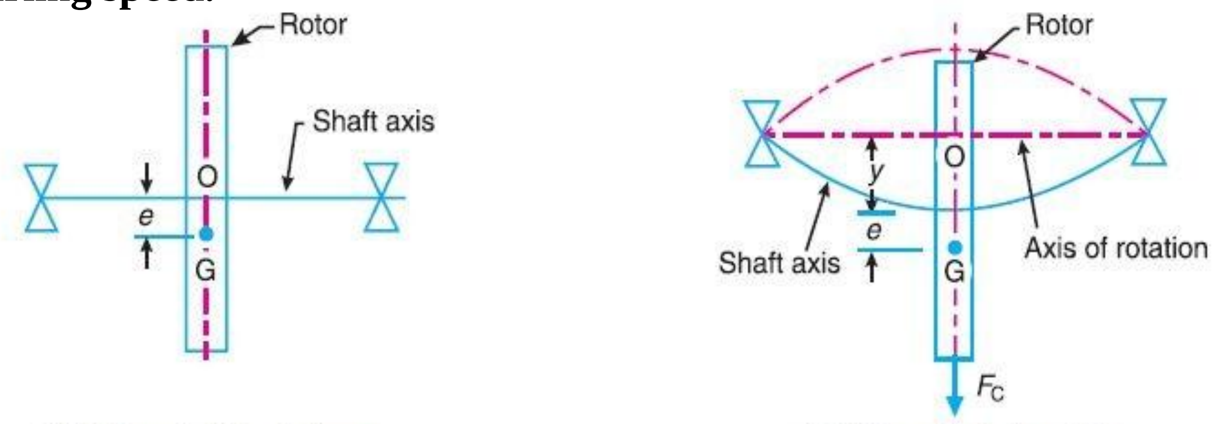
$$= \frac{0.4985}{0.1422} = 3.5 \text{ Hz}$$ **Ans.**

Critical or Whirling Speed of a Shaft

السرعات الحرجة او دوران المحاور

In actual practice, a rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc. When the gears or pulleys are put on the shaft, the Centre of gravity of the pulley or gear does not coincide with the Centre line of the bearings or with the axis of the shaft, when the shaft is stationary. This means that the Centre of gravity of the pulley or gear is at a certain distance from the axis of rotation and due to this, the shaft is subjected to centrifugal force. This force will bent the shaft which will further increase the distance of Centre of gravity of the pulley or gear from the axis of rotation. This correspondingly increases the value of centrifugal force, which further increases the distance of Centre of gravity from the axis of rotation. This effect is cumulative and ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity (distance between Centre of gravity of the pulley and the axis of rotation) but also depends upon the speed at which the shaft rotates.

The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as critical or whirling speed.



(*a*) When shaft is stationary.

(*b*) When shaft is rotating.

Fig .1 Critical or whirling speed of a shaft.

Consider a shaft of negligible mass carrying a rotor, as shown in Fig.1 (a). The point O is on the shaft axis and G is the centre of gravity of the rotor. When the shaft is stationary, the centre line of the bearing and the axis of the shaft coincides. Fig.1(b) shows the shaft when rotating about the axis of rotation at a uniform speed of ω rad/s

Let :

m = Mass of the rotor

e = Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary,

y = Additional deflection of centre of gravity of the rotor when the shaft starts rotating at ω rad/s, and

s = Stiffness of the shaft i.e. the load required per unit deflection of the shaft. Since the shaft is rotating at ω rad/s, therefore centrifugal force acting radially outwards through G causing the shaft to deflect is given by

$$F_C = m.\omega^2 (y + e)$$

The shaft behaves like a spring. Therefore the force resisting the deflection y,

$$= s.y$$

For the equilibrium position,

$$m.\omega^2 (y + e) = s.y$$

or $\quad m.\omega^2.y + m.\omega^2.e = s.y \quad$ or $\quad y(s - m.\omega^2) = m.\omega^2.e$

$$\therefore \quad y = \frac{m.\omega^2.e}{s - m.\omega^2} = \frac{\omega^2.e}{s/m - \omega^2} \qquad \ldots (i)$$

We know that circular frequency,

$$\omega_n = \sqrt{\frac{s}{m}} \quad \text{or} \quad y = \frac{\omega^2.e}{(\omega_n)^2 - \omega^2} \qquad \ldots \text{[From equation } (i) \text{]}$$

A little consideration will show that when $\omega > \omega_n$, the value of y will be negative and the shaft deflects is the opposite direction as shown dotted in Fig 23.14 (b).

In order to have the value of y always positive, both ***plus*** and ***minus*** signs are taken.

$$\therefore \quad y = \pm \frac{\omega^2 e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

... (Substituting $\omega_n = \omega_c$)

We see from the above expression that when $\omega_n = \omega_c$, the value of y becomes infinite.

Therefore ω_c is the **critical or whirling speed.**

$\therefore$ Critical or whirling speed,

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz} \qquad \ldots \left(\because \ \delta = \frac{m.g}{s}\right)$$

If N_c is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad \text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

where δ = Static deflection of the shaft in metres.

Hence the **critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.**

Notes :

1. When the centre of gravity of the rotor lies between the centre line of the shaft and the centre line of the bearing, e is taken negative. On the other hand, if the centre of gravity of the rotor does not lie between the centre line of the shaft and the centre line of the bearing (as in the above article) the value of e is taken positive.

2. To determine the critical speed of a shaft which may be subjected to point loads, uniformly distributed load or combination of both, find the frequency of transverse vibration which is equal to critical speed of a shaft in r.p.s. The Dunkerley's method may be used for calculating the frequency.

3. A shaft supported is short bearings (or ball bearings) is assumed to be a simply supported shaft while the shaft supported in long bearings (or journal bearings) is assumed to have both ends fixed.