a) Find the rate dy/dt (m/h) at which the tank is draining at time t.

b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?

c) GRAPHER Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

30. The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

a) At what rate does the volume change with respect to the radius when $r = 2$ ft?

b) By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

31. Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. If the aircraft will become airborne when its speed reaches 200 km/hr, how long will it take to become airborne, and what distance will it travel in that time?

32. *Volcanic lava fountains.* Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? in miles per hour? (*Hint:* If v_0 is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0 t - 16t^2$ feet. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

Grapher Explorations

Exercises 33–36 give the position function $s = f(t)$ of a body moving along the s-axis as a function of time t. Graph f together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the body's behavior in relation to the signs and values v and a. Include in your commentary such topics as the following.

a) When is the body momentarily at rest?

b) When does it move to the left (down) or to the right (up)?

c) When does it change direction?

d) When does it speed up and slow down?

e) When is it moving fastest (highest speed)? slowest?

f) When is it farthest from the axis origin?

33. $s = 200t - 16t^2, \quad 0 \le t \le 12.5$ (A heavy object fired straight up from the earth's surface at 200 ft/sec)

34. $s = t^2 - 3t + 2, \quad 0 \le t \le 5$

35. $s = t^3 - 6t^2 + 7t, \quad 0 \le t \le 4$

36. $s = 4 - 7t + 6t^2 - t^3, \quad 0 \le t \le 4$

2.4 Derivatives of Trigonometric Functions

Trigonometric functions are important because so many of the phenomena we want information about are periodic (electromagnetic fields, heart rhythms, tides, weather). A surprising and beautiful theorem from advanced calculus says that every periodic function we are likely to use in mathematical modeling can be written as an algebraic combination of sines and cosines, so the derivatives of sines and cosines play a key role in describing important changes. This section shows how to differentiate the six basic trigonometric functions.

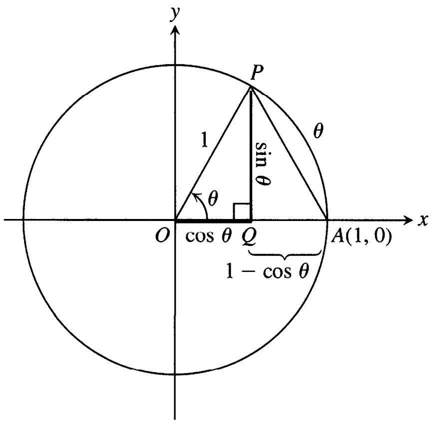
Some Special Limits

Our first step is to establish some inequalities and limits. It is assumed throughout that angles are measured in radians.

Theorem 3

If θ is measured in radians, then

$$-|\theta| < \sin\theta < |\theta| \quad \text{and} \quad -|\theta| < 1 - \cos\theta < |\theta|.$$



2.32 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1 - \cos\theta)^2 < \theta^2$.

Proof To establish these inequalities, we picture θ as an angle in standard position (Fig. 2.32). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 < \theta^2. \tag{1}$$

The terms on the left side of Eq. (1) are both positive, so each is smaller than their sum and hence is less than θ^2:

$$\sin^2\theta < \theta^2 \qquad \text{and} \qquad (1 - \cos\theta)^2 < \theta^2.$$

By taking square roots, we can see that this is equivalent to saying that

$$|\sin\theta| < |\theta| \qquad \text{and} \qquad |1 - \cos\theta| < |\theta|$$

or

$$-|\theta| < \sin\theta < |\theta| \qquad \text{and} \qquad -|\theta| < 1 - \cos\theta < |\theta|.$$

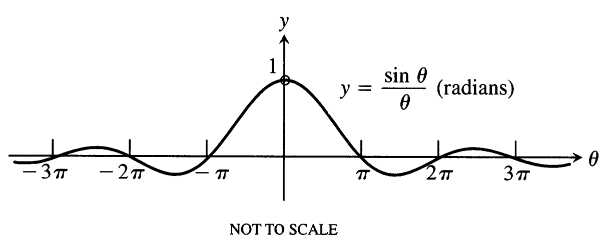
❑

EXAMPLE 1 Show that $\sin\theta$ and $\cos\theta$ are continuous at $\theta = 0$. That is,

$$\lim_{\theta \to 0} \sin\theta = 0 \qquad \text{and} \qquad \lim_{\theta \to 0} \cos\theta = 1.$$

Solution As $\theta \to 0$, both $|\theta|$ and $-|\theta|$ approach 0. The values of the limits therefore follow immediately from Theorem 3 and the Sandwich Theorem. ❑

The function $f(\theta) = (\sin\theta)/\theta$ graphed in Fig. 2.33 appears to have a removable discontinuity at $\theta = 0$. As the figure suggests, $\lim_{\theta \to 0} f(\theta) = 1$.



2.33 The graph of $f(\theta) = (\sin\theta)/\theta$.

Theorem 4

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{2}$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with values of θ that are positive and less than $\pi/2$ (Fig. 2.34). Notice that

$$\text{Area } \Delta OAP < \text{area sector } OAP < \text{area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

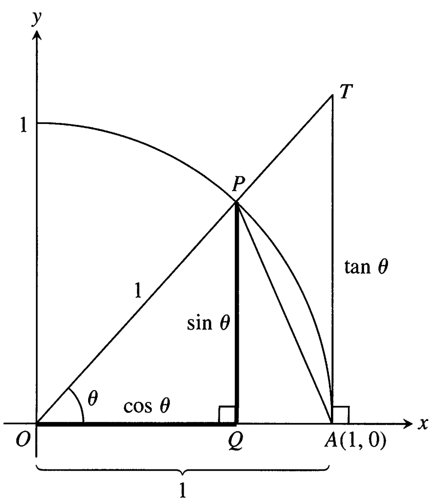
$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \tag{3}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta,$$

Equation (3) is where radian measure comes in: The area of sector *OAP* is $\theta/2$ only if θ is measured in radians.

so

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$



2.34 The figure for the proof of Theorem 4. *TA*/*OA* = $\tan\theta$, but *OA* = 1, so *TA* = $\tan\theta$.

This last inequality will go the same way if we divide all three terms by the positive number $(1/2)\sin\theta$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

We next take reciprocals, which reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

Finally, observe that $\sin\theta$ and θ are both *odd functions.* Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function,* with a graph symmetric about the y-axis (see Fig. 2.33). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 5 of Section 1.4. ❑

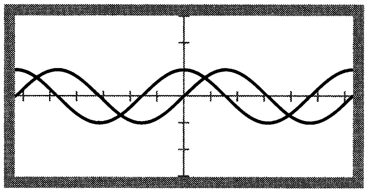
Theorem 4 can be combined with limit rules and known trigonometric identities to yield other trigonometric limits.

EXAMPLE 2 Show that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$

Solution Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \qquad \text{Let } \theta = h/2. \\ &= -(1)(0) = 0.\end{aligned}$$

❑



$y_1 = \sin x, -2\pi \le x \le 2\pi$
$y_2 = d(y_1)/dx, -2\pi \le x \le 2\pi$

Technology *Conjectures Based on Grapher Images* What you see in the window of a graphing utility can suggest conjectures, sometimes rather strongly. Graph the functions

$$y_1 = \sin x$$
$$y_2 = d(y_1)/dx \qquad \text{(This is computed by a built-in differentiation utility.)}$$

Does the graph of y_2 look familiar? What function do you think it is? Test your conjecture by adding the function's graph to the screen.

The Derivative of the Sine

To calculate the derivative of $y = \sin x$, we combine the limits in Example 2 and Theorem 4 with the addition formula

$$\sin(x + h) = \sin x \cos h + \cos x \sin h. \tag{4}$$

We have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Eq. (4)} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 2 and Theorem 4} \\
&= \cos x.
\end{aligned}$$

In short, the derivative of the sine is the cosine.

$$\frac{d}{dx}(\sin x) = \cos x$$

EXAMPLE 3

a) $y = x^2 - \sin x$:
$$\begin{aligned}
\frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\
&= 2x - \cos x
\end{aligned}$$

b) $y = x^2 \sin x$:
$$\begin{aligned}
\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x && \text{Product Rule} \\
&= x^2 \cos x + 2x \sin x
\end{aligned}$$

c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \quad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}$$ ❑

Radian measure in calculus

In case you are wondering why calculus uses radian measure when the rest of the world seems to use degrees, the answer lies in the argument that the derivative of the sine is the cosine. The derivative of $\sin x$ is $\cos x$ *only* if x is measured in radians. The argument requires that when h is a small increment in x,

$$\lim_{h \to 0} (\sin h)/h = 1.$$

This is true only for radian measure, as we saw during the proof of Theorem 4. You will see what the degree-mode derivatives of the sine and cosine are if you do Exercise 76.

The Derivative of the Cosine

With the help of the addition formula,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h, \tag{5}$$

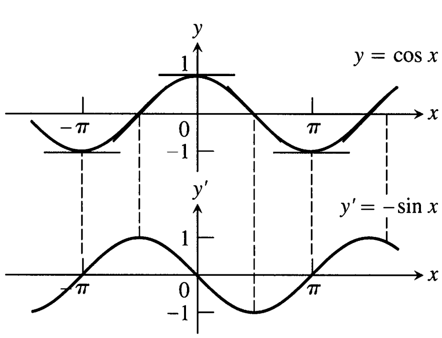
we have

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} && \text{Derivative definition} \\
&= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Eq. (5)} \\
&= \lim_{h \to 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 && \text{Example 2 and Theorem 4} \\
&= -\sin x.
\end{aligned}$$

In short, the derivative of the cosine is the negative of the sine.

$$\frac{d}{dx}(\cos x) = -\sin x$$

Figure 2.35 shows another way to visualize this result.



2.35 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

EXAMPLE 4

a) $y = 5x + \cos x$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) && \text{Sum Rule} \\
&= 5 - \sin x
\end{aligned}$$

b) $y = \sin x \cos x$

$$\begin{aligned}
\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule} \\
&= \sin x(-\sin x) + \cos x(\cos x) \\
&= \cos^2 x - \sin^2 x
\end{aligned}$$

c) $$y = \frac{\cos x}{1 - \sin x}$$

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

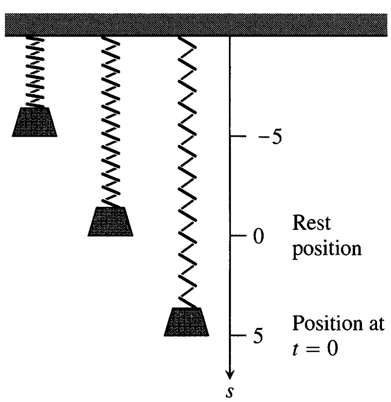
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}$$

❑

Simple Harmonic Motion

The motion of a body bobbing up and down on the end of a spring is an example of *simple harmonic motion.* The next example describes a case in which there are no opposing forces like friction or buoyancy to slow the motion down.



2.36 The body in Example 5.

EXAMPLE 5 A body hanging from a spring (Fig. 2.36) is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5\cos t.$$

What are its velocity and acceleration at time t?

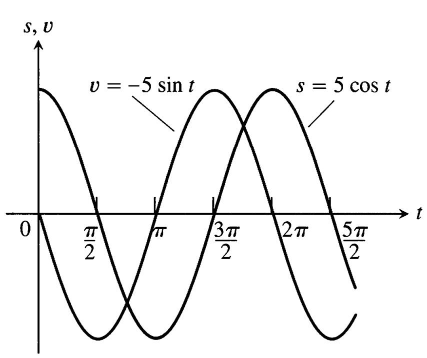
Solution We have

Position: $s = 5\cos t$

Velocity: $v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = 5\frac{d}{dt}(\cos t) = -5\sin t$

Acceleration: $a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\frac{d}{dt}(\sin t) = -5\cos t.$

Here is what we can learn from these equations:



2.37 The graphs of the position and velocity of the body in Example 5.

1. As time passes, the body moves up and down between $s = 5$ and $s = -5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π, the period of $\cos t$.
2. The function $\sin t$ attains its greatest magnitude (1) when $\cos t = 0$, as the graphs of the sine and cosine show (Fig. 2.37). Hence, the body's speed, $|v| = 5|\sin t|$, is greatest every time $\cos t = 0$, i.e., every time the body passes its rest position.

   The body's speed is zero when $\sin t = 0$. This occurs at the endpoints of the interval of motion, when $\cos t = \pm 1$.
3. The acceleration, $a = -5\cos t$, is zero only at the rest position, where the cosine is zero. When the body is anywhere else, the spring is either pulling on it or pushing on it. The acceleration is greatest in magnitude at the points farthest from the origin, where $\cos t = \pm 1$. ❑

Jerk

A sudden change in acceleration is called a "jerk." When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt. Jerk is what spills your soft drink. The derivative responsible for jerk is d^3s/dt^3.

Definition

Jerk is the derivative of acceleration. If a body's position at time t is $s = f(t)$, the body's jerk at time t is

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Recent tests have shown that motion sickness comes from accelerations whose changes in magnitude or direction take us by surprise. Keeping an eye on the road helps us to see the changes coming. A driver is less likely to become sick than a passenger reading in the backseat.

EXAMPLE 6

a) The jerk of the constant acceleration of gravity ($g = 32$ ft/sec^2) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

We don't experience motion sickness if we are just sitting around.

b) The jerk of the simple harmonic motion in Example 5 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t)$$
$$= 5\sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the origin, where the acceleration changes direction and sign. ❑

The Derivatives of the Other Basic Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas.

Notice the minus signs in the derivative formulas for the cofunctions.

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad (6) \qquad \frac{d}{dx}(\sec x) = \sec x \tan x \quad (7)$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \quad (8) \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \quad (9)$$

To show how a typical calculation goes, we derive Eq. (6). The other derivations are left to Exercises 67 and 68.

EXAMPLE 7 Find dy/dx if $y = \tan x$.

Solution

$$\begin{aligned}\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule}\\ &= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

❑

EXAMPLE 8 Find y'' if $y = \sec x$.

Solution

$$\begin{aligned}y &= \sec x\\ y' &= \sec x \tan x && \text{Eq. (7)}\\ y'' &= \frac{d}{dx}(\sec x \tan x)\\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Product Rule}\\ &= \sec x\,(\sec^2 x) + \tan x\,(\sec x \tan x)\\ &= \sec^3 x + \sec x \tan^2 x\end{aligned}$$

❑

EXAMPLE 9

a) $\dfrac{d}{dx}(3x + \cot x) = 3 + \dfrac{d}{dx}(\cot x) = 3 - \csc^2 x$

b) $\dfrac{d}{dx}\left(\dfrac{2}{\sin x}\right) = \dfrac{d}{dx}(2\csc x) = 2\dfrac{d}{dx}(\csc x)$

$= 2(-\csc x \cot x) = -2\csc x \cot x$

❑

Continuity of Trigonometric Functions

Since the six basic trigonometric functions are differentiable throughout their domains they are also continuous throughout their domains by Theorem 1, Section 2.1. This means that $\sin x$ and $\cos x$ are continuous for all x, that $\sec x$ and $\tan x$ are continuous except when x is a nonzero integer multiple of $\pi/2$, and that $\csc x$ and $\cot x$ are continuous except when x is an integer multiple of π. For each function, $\lim_{x\to c} f(x) = f(c)$ whenever $f(c)$ is defined. As a result, we can calculate the limits of many algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 10

$$\lim_{x\to 0} \frac{\sqrt{2+\sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+\sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$ ❑

Other Limits Calculated with Theorem 4

The equation $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ holds no matter how θ may be expressed:

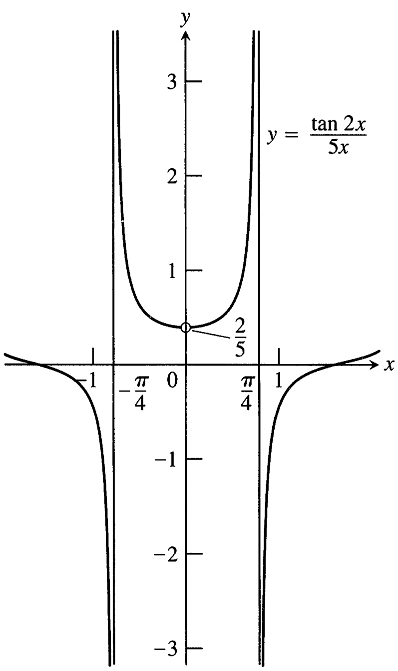
$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \theta = x; \qquad \lim_{x\to 0}\frac{\sin 7x}{7x} = 1, \quad \theta = 7x;$$

As $x \to 0$, $\theta \to 0$ As $x \to 0$, $\theta \to 0$

$$\lim_{x\to 0}\frac{\sin(2/3)x}{(2/3)x} = 1, \quad \theta = (2/3)x$$

As $x \to 0$, $\theta \to 0$

Knowing this helps us calculate related limits involving angles in radian measure.



2.38 The graph of $y = (\tan 2x)/5x$ steps across the y-axis at $y = 2/5$ (Example 11).

EXAMPLE 11

a) $$\lim_{x\to 0}\frac{\sin 2x}{5x} = \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x}$$

Eq. (2) does not apply to the original fraction. We need $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by 2/5.

$$= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}$$

Now Eq. (2) applies

$$= \frac{2}{5}(1) = \frac{2}{5}$$

b) $$\lim_{x\to 0}\frac{\tan 2x}{5x} = \lim_{x\to 0}\left(\frac{\sin 2x}{5x}\cdot\frac{1}{\cos 2x}\right)$$

$\tan 2x = \frac{\sin 2x}{\cos 2x}$

$$= \left(\lim_{x\to 0}\frac{\sin 2x}{5x}\right)\left(\lim_{x\to 0}\frac{1}{\cos 2x}\right)$$

$$= \left(\frac{2}{5}\right)\left(\frac{1}{\cos 0}\right) = \frac{2}{5}$$

Part (a) and continuity of $\cos x$

See Fig. 2.38. ❑

Applications

The occurrence of the function $(\sin x)/x$ in calculus is not an isolated event. The function arises in such diverse fields as quantum physics (where it appears in solutions of the wave equation) and electrical engineering (in signal analysis and signal filter design) as well as in the mathematical fields of differential equations and probability theory.

EXAMPLE 12

$$\lim_{t\to(\pi/2)} \frac{\sin\left(t-\frac{\pi}{2}\right)}{t-\frac{\pi}{2}}$$

Set $\theta = t - (\pi/2)$. Then $\theta \to 0$ as $t \to (\pi/2)$.

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

❑

Exercises 2.4

Derivatives

In Exercises 1–12, find dy/dx.

1. $y = -10x + 3\cos x$

2. $y = \frac{3}{x} + 5\sin x$

3. $y = \csc x - 4\sqrt{x} + 7$

4. $y = x^2 \cot x - \frac{1}{x^2}$

5. $y = (\sec x + \tan x)(\sec x - \tan x)$

6. $y = (\sin x + \cos x)\sec x$

7. $y = \frac{\cot x}{1+\cot x}$

8. $y = \frac{\cos x}{1+\sin x}$

9. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

10. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

11. $y = x^2 \sin x + 2x\cos x - 2\sin x$

12. $y = x^2 \cos x - 2x\sin x - 2\cos x$

In Exercises 13–16, find ds/dt.

13. $s = \tan t - t$

14. $s = t^2 - \sec t + 1$

15. $s = \frac{1+\csc t}{1-\csc t}$

16. $s = \frac{\sin t}{1-\cos t}$

In Exercises 17–20, find $dr/d\theta$.

17. $r = 4 - \theta^2 \sin\theta$

18. $r = \theta\sin\theta + \cos\theta$

19. $r = \sec\theta\csc\theta$

20. $r = (1+\sec\theta)\sin\theta$

In Exercises 21–24, find dp/dq.

21. $p = 5 + \frac{1}{\cot q}$

22. $p = (1+\csc q)\cos q$

23. $p = \frac{\sin q + \cos q}{\cos q}$

24. $p = \frac{\tan q}{1+\tan q}$

25. Find y'' if (a) $y = \csc x$, (b) $y = \sec x$.

26. Find $y^{(4)} = d^4y/dx^4$ if (a) $y = -2\sin x$, (b) $y = 9\cos x$.

Limits

Find the limits in Exercises 27–32.

27. $\lim_{x\to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

28. $\lim_{x\to -\pi/6} \sqrt{1+\cos(\pi \csc x)}$

29. $\lim_{x\to 0} \sec\left[\cos x + \pi\tan\left(\frac{\pi}{4\sec x}\right) - 1\right]$

30. $\lim_{x\to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2\sec x}\right)$

31. $\lim_{t\to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

32. $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

Find the limits in Exercises 33–48.

33. $\lim_{\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

34. $\lim_{t\to 0} \frac{\sin kt}{t}$ (k constant)

35. $\lim_{y\to 0} \frac{\sin 3y}{4y}$

36. $\lim_{h\to 0^-} \frac{h}{\sin 3h}$

37. $\lim_{x\to 0} \frac{\tan 2x}{x}$

38. $\lim_{t\to 0} \frac{2t}{\tan t}$

39. $\lim_{x\to 0} \frac{x\csc 2x}{\cos 5x}$

40. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

41. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x\cos x}$

42. $\lim_{x\to 0} \frac{x^2 - x + \sin x}{2x}$

43. $\lim_{t\to 0} \frac{\sin(1-\cos t)}{1-\cos t}$

44. $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h}$

45. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$

46. $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$

47. $\lim_{x\to 0} \dfrac{\tan 3x}{\sin 8x}$

48. $\lim_{y\to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y}$

Tangent Lines

In Exercises 49–52, graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

49. $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$
$x = -\pi,\ 0,\ 3\pi/2$

50. $y = \tan x, \quad -\pi/2 < x < \pi/2$
$x = -\pi/3,\ 0,\ \pi/3$

51. $y = \sec x, \quad -\pi/2 < x < \pi/2$
$x = -\pi/3,\ \pi/4$

52. $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$
$x = -\pi/3,\ 3\pi/2$

Do the graphs of the functions in Exercises 53–56 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not? You may want to visualize your findings by graphing the functions with a grapher.

53. $y = x + \sin x$

54. $y = 2x + \sin x$

55. $y = x - \cot x$

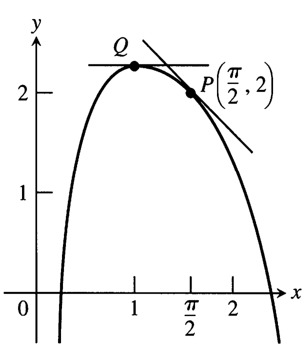
56. $y = x + 2\cos x$

57. Find all points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

58. Find all points on the curve $y = \cot x, 0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

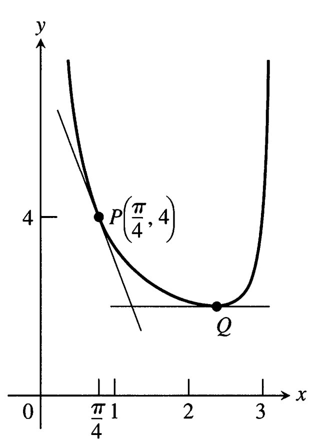
In Exercises 59 and 60, find an equation for (a) the tangent to the curve at P and (b) the horizontal tangent to the curve at Q.

59.



$y = 4 + \cot x - 2\csc x$
(Generated by Mathematica)

60.



$y = 1 + \sqrt{2}\csc x + \cot x$
(Generated by Mathematica)

Simple Harmonic Motion

The equations in Exercises 61 and 62 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

61. $s = 2 - 2\sin t$

62. $s = \sin t + \cos t$

Theory and Examples

63. Is there a value of c that will make

$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \ne 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

64. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \ge 0 \end{cases}$$

continuous at $x = 0$? differentiable at $x = 0$? Give reasons for your answers.

65. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$

66. Find $\dfrac{d^{725}}{dx^{725}}(\sin x)$

67. Derive the formula for the derivative with respect to x of

a) $\sec x$ **b)** $\csc x$.

68. Derive the formula for the derivative with respect to x of $\cot x$.

Grapher Explorations

69. Graph $y = \cos x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\sin(x+h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? as $h \to 0^-$? What phenomenon is being illustrated here?

70. Graph $y = -\sin x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\cos(x+h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? as $h \to 0^-$? What phenomenon is being illustrated here?

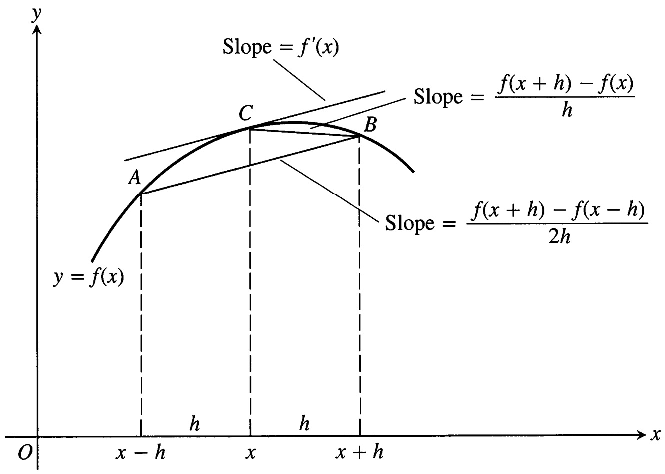
71. *Centered difference quotients.* The **centered difference quotient**

$$\frac{f(x+h) - f(x-h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than Fermat's difference quotient

$$\frac{f(x+h) - f(x)}{h}.$$

See the figure below.



a) To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 69 for the same values of h.

b) To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 70 for the same values of h.

72. *A caution about centered difference quotients. (Continuation of Exercise 71.)* The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h\to 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$.

73. Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? a largest slope? Is the slope ever negative? Give reasons for your answers.

74. Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? a largest slope? Is the slope ever positive? Give reasons for your answers.

75. Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the y-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k? Give reasons for your answers.

76. *Radians vs. degrees.* What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

a) With your graphing calculator or computer grapher *in degree mode,* graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h\to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

b) With your grapher still in degree mode, estimate

$$\lim_{h\to 0} \frac{\cos h - 1}{h}.$$

c) Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

d) Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?

e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

2.5 The Chain Rule

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

Section 2.4, pp. 152–154

1. $-10 - 3 \sin x$ **3.** $-\csc x \cot x - \dfrac{2}{\sqrt{x}}$ **5.** 0

7. $\dfrac{-\csc^2 x}{(1 + \cot x)^2}$ **9.** $4 \tan x \sec x - \csc^2 x$ **11.** $x^2 \cos x$

13. $\sec^2 t - 1$ **15.** $\dfrac{-2 \csc t \cot t}{(1 - \csc t)^2}$ **17.** $-\theta\,(\theta \cos \theta + 2 \sin \theta)$

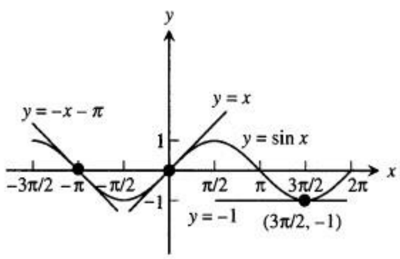
19. $\sec \theta \csc \theta\,(\tan \theta - \cot \theta) = \sec^2 \theta - \csc^2 \theta$ **21.** $\sec^2 q$

23. $\sec^2 q$ **25.** a) $2 \csc^3 x - \csc x$ b) $2 \sec^3 x - \sec x$

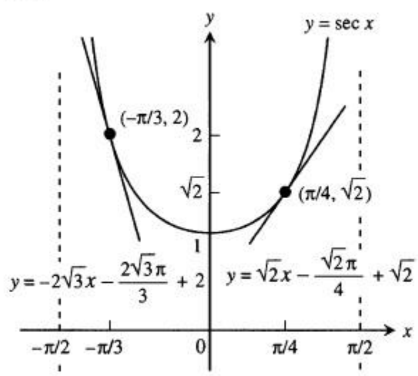
27. 0 **29.** -1 **31.** 0 **33.** 1 **35.** 3/4 **37.** 2 **39.** 1/2

41. 2 **43.** 1 **45.** 1/2 **47.** 3/8

49.

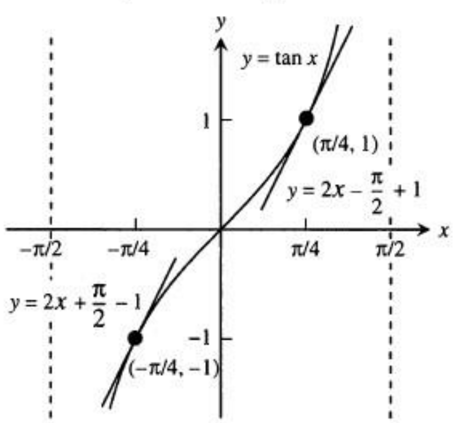


51.



53. Yes, at $x = \pi$ **55.** No

57. $\left(-\dfrac{\pi}{4}, -1\right); \left(\dfrac{\pi}{4}, 1\right)$



59. a) $y = -x + \pi/2 + 2$ b) $y = 4 - \sqrt{3}$

61. $-\sqrt{2}$ m/sec, $\sqrt{2}$ m/sec, $\sqrt{2}$ m/sec^2, $\sqrt{2}$ m/sec^3 **63.** $c = 9$

65. $\sin x$