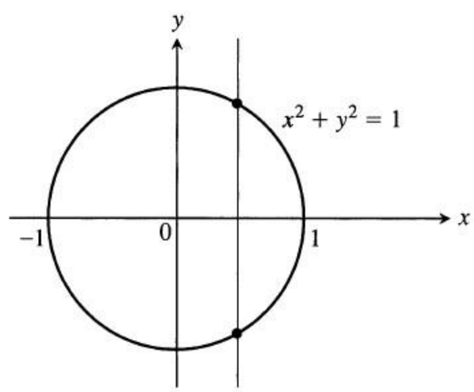
Function and Graphs

Graphs of Functions

The **graph** of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane whose coordinates (x, y) are input–output pairs for f.

Not every curve you draw is the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no ***vertical line*** can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function since some vertical lines intersect the circle twice (Fig. 25). If a is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f in the single point $(a, f(a))$.



25 This circle is not the graph of a function $y = f(x)$; it fails the vertical line test.

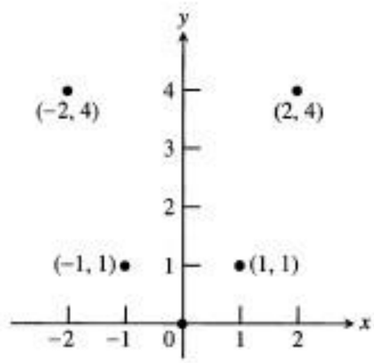
EXAMPLE 4 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution

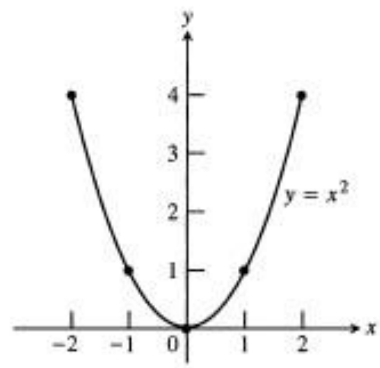
x	$y = x^2$
−2	4
−1	1
0	0
1	1
2	4

Step 1: Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

Step 2: Plot the points (x, y) whose coordinates appear in the table.

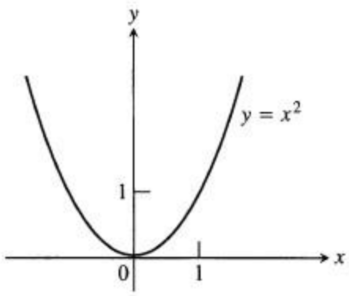


Step 3: Draw a smooth curve through the plotted points. Label the curve with its equation.

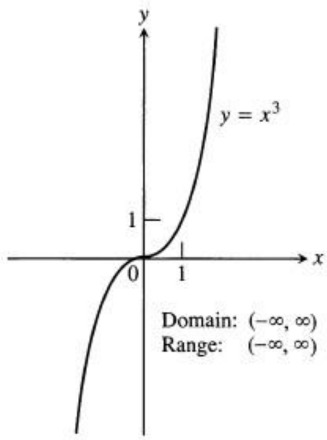


❑

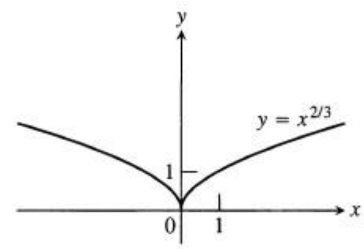
Figure 26 shows the graphs of several functions frequently encountered in calculus. It is a good idea to learn the shapes of these graphs so that you can recognize them or sketch them when the need arises.



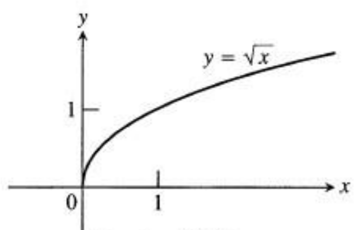
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



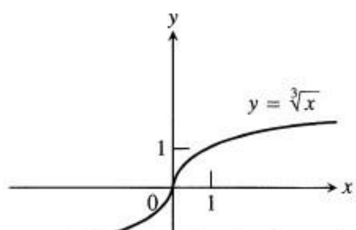
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



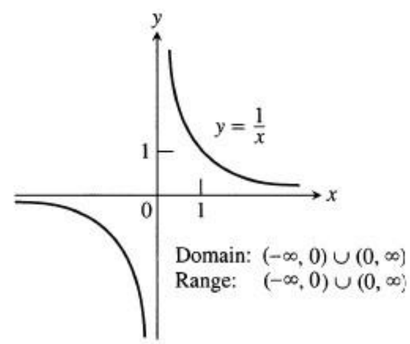
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



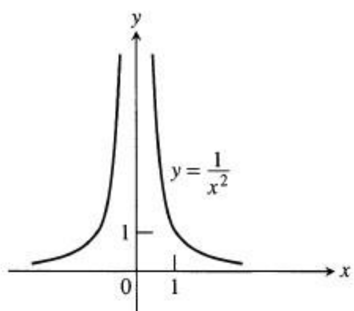
Domain: $[0, \infty)$
Range: $[0, \infty)$



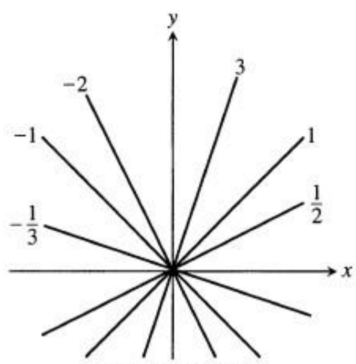
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



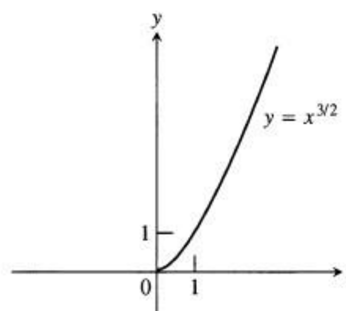
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$



$y = mx$ for selected values of m
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



Domain: $[0, \infty)$
Range: $[0, \infty)$

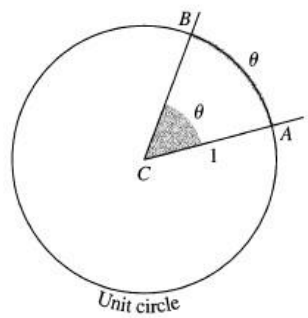
5 Trigonometric Functions

This section reviews radian measure, trigonometric functions, periodicity, and basic trigonometric identities.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations (Section 2.4).

Let ACB be a central angle in a **unit circle** (circle of radius 1), as in Fig. 47.



47 The radian measure of angle *ACB* is the length of the arc *AB*.

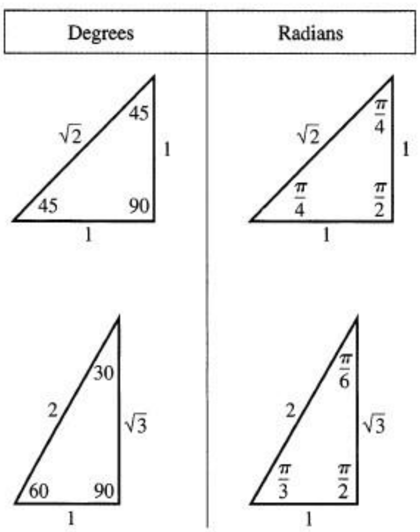
The **radian measure** $\boldsymbol{\theta}$ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by the following equation.

$$\pi \text{ radians} = 180^\circ$$

EXAMPLE 1 *Conversions (Fig. 48)*

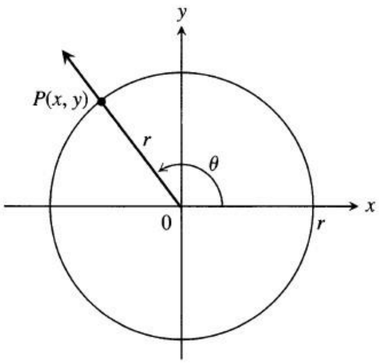
Convert 45° to radians: $45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad

Convert $\dfrac{\pi}{6}$ rad to degrees: $\dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30^\circ$ ❑

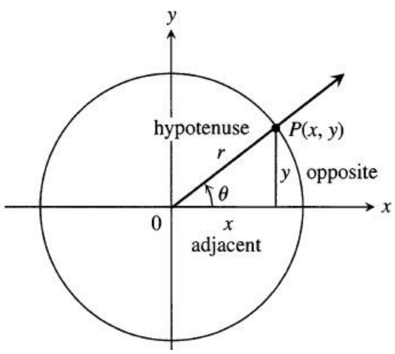


48 The angles of two common triangles, in degrees and radians.

Arc Length



53 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.



EXAMPLE 2 Consider a circle of radius 8. (a) Find the central angle subtended by an arc of length 2π on the circle. (b) Find the length of an arc subtending a central angle of $3\pi/4$.

Solution

a) $\theta = \dfrac{s}{r} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$ **b)** $s = r\theta = 8\left(\dfrac{3\pi}{4}\right) = 6\pi$ ❑

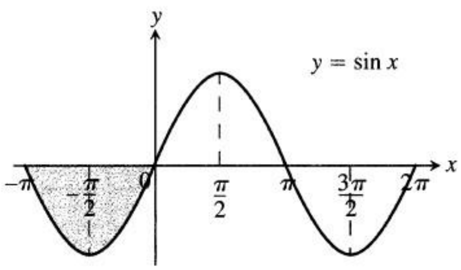
The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Fig. 52). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Fig. 53).

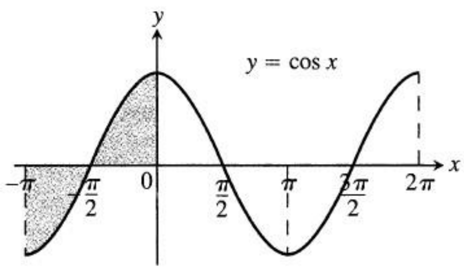
Sine:	$\sin\theta = \dfrac{y}{r}$	Cosecant:	$\csc\theta = \dfrac{r}{y}$
Cosine:	$\cos\theta = \dfrac{x}{r}$	Secant:	$\sec\theta = \dfrac{r}{x}$
Tangent:	$\tan\theta = \dfrac{y}{x}$	Cotangent:	$\cot\theta = \dfrac{x}{y}$

Table 2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

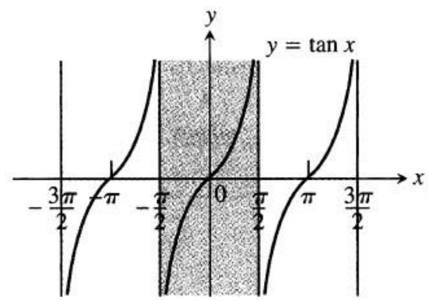
Degrees	−180	−135	−90	−45	0	30	45	60	90	135	180
θ (radians)	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π
$\sin\theta$	0	$-\sqrt{2}/2$	−1	$-\sqrt{2}/2$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{2}/2$	0
$\cos\theta$	−1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	$-\sqrt{2}/2$	−1
$\tan\theta$	0	1		−1	0	$\sqrt{3}/3$	1	$\sqrt{3}$		−1	0



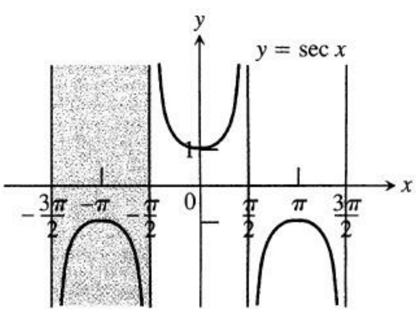
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



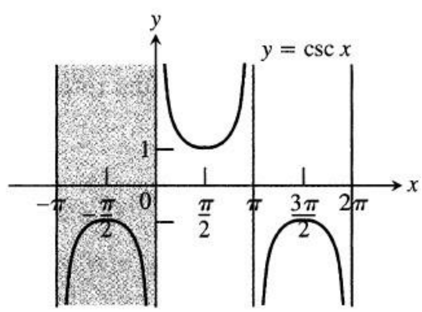
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



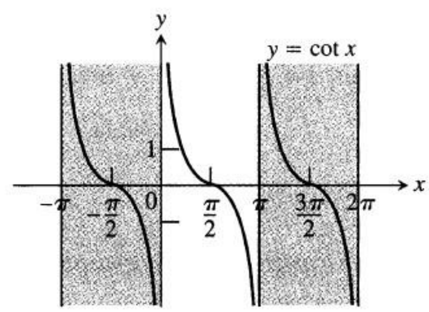
Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, \infty)$



Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, \infty)$

Basic relationships

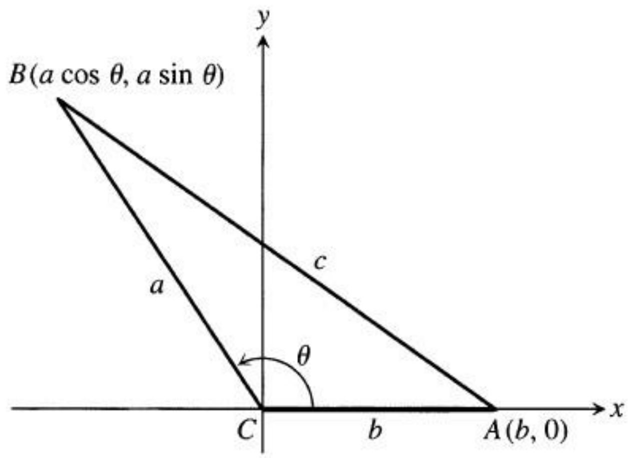
$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta,$$
$$1 + \cot^2\theta = \csc^2\theta.$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$



The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \qquad (7)$$