Chain Rule :

The chain rule is used to differentiate the composite functions.

1. Chain rule for function of single variable defined along paths.

$$y = f(x)$$

$$X = x(t) \quad \Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx} * \frac{dx}{dt}$$

$$y = F[x(t)]$$

Ex: if $y = x^2 + 1$, $X = \tan^{-1} t$, find $\frac{dy}{dt}$?

Sol: $\frac{dy}{dx} = 2x$

$$\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= 2x \cdot \frac{1}{1+t^2} = 2\tan^{-1} t * \frac{1}{1+t^2}$$

2. For the function $Z = f(x,y)$ of two variable defined along path.

$$Z = f[x(t), g(t)]$$

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} + \frac{dy}{dt}$$

Ex: $Z = f(x,y) = x^2 y^3$, $x = \cos t$, $y = \sin t$ find $\frac{df}{dt}$?

Sol:

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

$$f_x = y^3 2x = 2xy^3$$

$$f_y = 3y^2 x^2$$

$$\frac{dy}{dt} = \cos t \quad , \quad \frac{dx}{dt} = -\sin t$$

$$\frac{df}{dt} = 2xy^3(-\sin t) + 3y^2x^2(\cos t)$$

$$= 2\cos t(\sin t)^3(-\sin t) + 3(\sin t)^2(\cos t)^2(\cos t)$$

$$= -2\cos t(\sin t)^4 + 3(\sin t)^2(\cos t)^3$$

3. chain rule for function to more than two variable defined along path :

$$w = f(x_1, x_2, x_3, ---, x_n)$$

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \cdots + \frac{df}{dx_n} \cdot \frac{dx_n}{dt}$$

Ex : let $W = xy \sin Z$ where $X = \cos t$, $y = \sin t$, $Z = 1 + t^2$

find $\frac{dw}{dt}$?

Sol:

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \frac{df}{dx_3} \cdot \frac{dx_3}{dt}$$

$F_x = y \sin Z$, $F_y = X \sin Z$, $F_z = Xy \cos Z$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{dz}{dt} = 2t$$

$$\frac{dw}{dt} = y \sin Z (-\sin t) + X \sin Z (\cos t) + Xy \cos Z (2t)$$

$$= y \sin(1+t^2)(-\sin t) + \cos t \sin(1+t^2) X + \cos t \sin t \cos(1+t^2)$$

$$(2t)$$

4. Chain rule for function of two variable defined on surface :

If $z = f(x,y)$ has partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and if

$x = x(r,s)$ and $y = y(r,s)$

$$z = f[x(r,s), y(r,s)]$$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Ex : find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ if $z = f(x,y) = x^2 + y^2$

$x = r + e^s$, $y = \ln s$?

Sol :

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$f_x = 2x$, $\frac{\partial x}{\partial r} = 1$, $\frac{\partial y}{\partial r} = 0$

$f_y = 2y$, $\frac{\partial x}{\partial s} = e^s$, $\frac{\partial y}{\partial s} = \frac{1}{s}$

$$\frac{\partial z}{\partial r} = 2x(1) + 2y(0) = 2x = 2(r + e^s)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2xe^{s} + 2y\,(1/s)$$

$$= 2\,(r+e^{s})\cdot e^{s} + 2\,(\ln s)\cdot \frac{1}{s}$$

$$= 2re^{s} + 2e^{2s} + \frac{2}{s}\ln s$$

Ex: If Z is a differentiable function of x and y which satisfy the equation $x^3 + y^3 + z^3 + 3x^2 \sin y \tan Z = 5$?

find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$?

Sol:

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 3\sin y\left(x^2 \sec^2 z\,\frac{\partial z}{\partial x} + \tan Z * 2x\right) = 0$$

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 3x^2 \sin y \sec^2 Z\,\frac{\partial z}{\partial x} + 6x \sin y \tan Z = 0$$

$$3x^2 \sin y \sec^2 Z\,\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = -3x^2 - 6x \sin y \tan Z$$

$$\frac{\partial z}{\partial x}\left(3x^2 \sin y \sec^2 Z + 3z^2\right) = -3x^2 - 6x \sin y \tan Z$$

$$\frac{\partial Z}{\partial x} = \frac{-3x^2 - 6x \sin y \tan Z}{3x^2 \sin y \sec^2 Z + 3Z^2}$$

$$0+3y^2+3z^2\frac{\partial z}{\partial y}+3x^2\left(\sin y\ \sec^2 z\ \frac{\partial z}{\partial y}+\tan z\ \cos y=0\right.$$

$$3y^2+3z^2\frac{\partial z}{\partial y}+3x^2\sin y\ \sec^2 z\ \frac{\partial z}{\partial y}+3x^2\tan z\ \cos y=0$$

$$3z^2\frac{\partial z}{\partial y}+3x^2\sin y\ \sec^2 z\ \frac{\partial z}{\partial y}=-3y^2-3x^2\tan z\ \cos y$$

$$\frac{\partial z}{\partial y}\left(3z^2+3x^2\sin y\ \sec^2 z\right)=-3y^2-3x^2\tan z\ \cos y$$

$$\frac{\partial z}{\partial y}=\frac{-3y^2-3x^2\tan z\ \cos y}{3z^2+3x^2\sin y\ \sec^2 z}$$

5. chain rule for function of three variables defined on surface :-

if $w=f(x,y,z)$ has partial derivatives

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ and if $x=x(r,s)$, $z=z(r,s)$

also have partial derivatives

$$w=f\left[x(r,s),\,y(r,s),\,z(r,s)\right]$$

$$\frac{\partial w}{\partial r}=\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Ex: find $\frac{\partial w}{\partial r}$ if $w = f(x, y, z) = x + 2y + z^2$ where $x = \frac{r}{s}$

$y = r^2 + e^s$, $z = 2r$?

Sol:

$$w = x + 2y + z^2$$

$$\frac{\partial w}{\partial x} = 1 + 0 + 0 = 1$$

$$\frac{\partial w}{\partial y} = 0 + 2 + 0 = 2$$

$$\frac{\partial w}{\partial z} = 0 + 0 + 2z = 2z$$

$$x = \frac{r}{s} \Rightarrow \frac{\partial x}{\partial r} = \frac{1}{s} * 1 = \frac{1}{s}$$

$$y = r^2 + e^s \Rightarrow \frac{\partial y}{\partial r} = 2r + 0 = 2r$$

$$z = 2r \Rightarrow \frac{\partial z}{\partial r} = 2$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= 1 * \frac{1}{s} + 2 * 2r + 2z * 2$$

$$= \frac{1}{s} + 4r + 4z$$

$$= \frac{1}{s} + 4r + 4(2r) = \frac{1}{s} + 12r$$