Differentiation   المشتقة

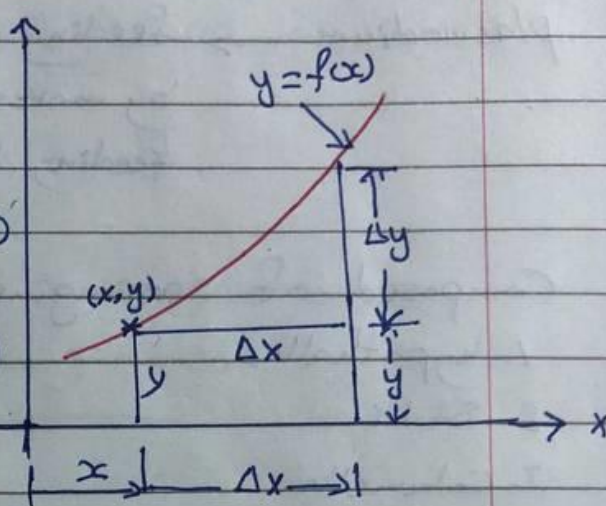
$$\cancel{y} = f(x) \quad \text{——} \ (1)$$

at $x + \Delta x$

$$\cancel{y} + \Delta y = f(x + \Delta x) \quad \text{——} \ (2)$$

2 طرح

$$\Delta y = f(x + \Delta x) - f(x)$$



divid by Δx to obtain

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example: Derive the function $y = x^2$

$$y = x^2 \quad \text{——} \ (1)$$

$$y + \Delta y = (x + \Delta x)^2$$

subtract

$$y + \Delta y - y = (x + \Delta x)^2 - x^3$$

$$\cancel{y} + \Delta y - \cancel{y} = \cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x^2}$$

$$\Delta y = 2x\Delta x + \Delta x^2$$

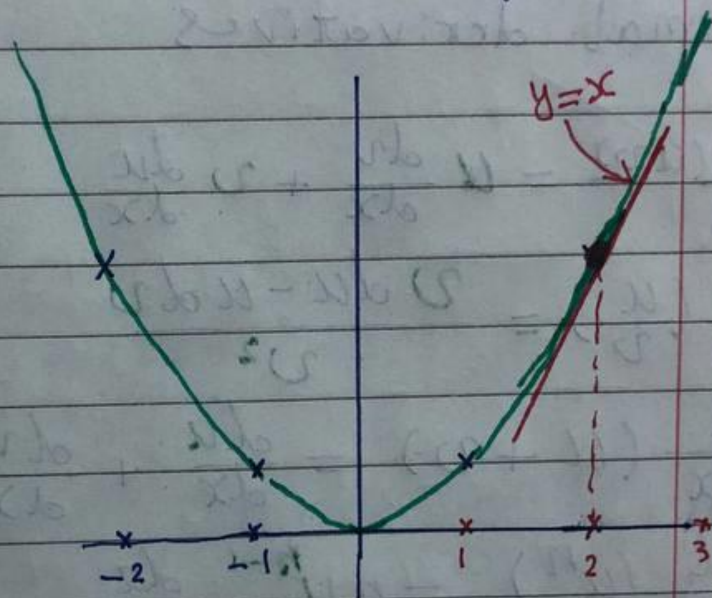
divid both side by Δx

$$\frac{\Delta y}{\Delta x} = \frac{2x\cancel{\Delta x}}{\cancel{\Delta x}} + \frac{\Delta x^{\cancel{2}}}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \cancel{\Delta x}^{\to 0})$$

$$\frac{dy}{dx} = 2x$$

Ex: Find the equation of a line tangent to the function ~~at~~ $y = x^2$ at a point $x = 2$



Ans:

$$y = x^2$$

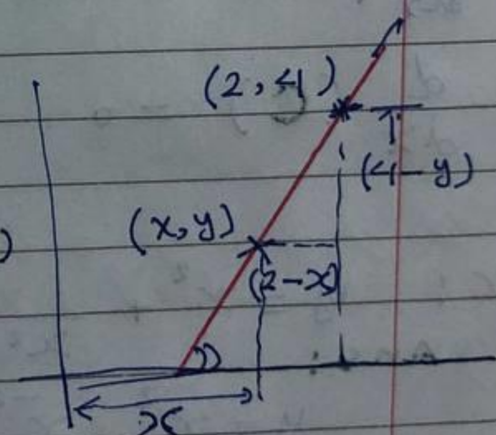
$$\frac{dy}{dx} = 2x$$

$$\therefore m = \frac{dy}{dx} = 2 * 2$$
$$= 4$$

$$y = 2^2 = 4$$

$$m = \frac{4-y}{2-x}$$

$$4 = \frac{4-y}{2-x}$$ (معادلة مستقيم)



$$4 - y = 4(2-x)$$
$$4 - y = 8 - 4x$$
$$-y = 8 - 4 - 4x$$
$$-y = 4 - 4x$$
$$y = 4x - 4$$

Ex: Find $\frac{dy}{dx}$ if $y = x^3 + 7x^2 - 5x + 4$

Ans: $\frac{dy}{dx} = 3x^2 + 14x - 5$

Formal derivatives

1) $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

2) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

3) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

4) $\frac{d}{dx}(u^n) = n\,u^{n-1}\frac{du}{dx}$

5) $\frac{d}{dx}(C) = 0$ where C: Constant

Ex: $y = x^2 + \frac{1}{x^2}$, $x \neq 0$

Ans:

$y = x^2 + x^{-2}$

$\therefore \frac{dy}{dx} = 2x - 2x^{-3}$

Problems: Find $\frac{dy}{dx}$

1) $y = (x^2+1)^5$

Ans:

$$\frac{dy}{dx} = 5(x^2+1)^4 * 2x \quad = 10x(x^2+1)^4$$

2) $y = \frac{2x+5}{3x-2}$

Ans:

$$\frac{dy}{dx} = \frac{(3x-2)*2 - (2x+5)*3}{(3x-2)^2}$$

$$= \frac{\cancel{6x} - 4 - \cancel{6x} - 15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$$

3) $y = (x-1)(x+2)$

Ans:

$$\frac{dy}{dx} = (x-1)*1 + (x+2)*1$$

$$= x-1 + x+2$$

$$= 2x+1$$

Find $\frac{ds}{dt}$ in each of the following problems

$$s = \frac{t}{t^2+1}$$

Ans: $\frac{ds}{dt} = \frac{(t^2+1) - t*2t}{(t^2+1)^2} = \frac{t^2+1-2t^2}{(t^2+1)} = \frac{(1-t^2)}{(1+t^2)}$

Ex: Find $\frac{dy}{dx}$ if $x^5 + 4xy^3 - 3y^5 = 2$

Ans:

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(4xy^3) - \frac{d}{dx}(3y^5) = \frac{d}{dx}(2)$$

$$5x^4 \cancel{\frac{dx}{dx}} + 4\left(x * 3y^2 \frac{dy}{dx} + y^3 \cancel{\frac{dx}{dx}}\right) - 3 * 5y^4 \frac{dy}{dx} = 0$$

$$5x^4 + 12xy^2 \frac{dy}{dx} + 4y^3 - 15y^4 \frac{dy}{dx} = 0$$

$$5x^4 + 4y^3 = (-12xy^2 + 15y^4)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{5x^4 + 4y^3}{-12xy^2 + 15y^4}$$

Ex Find $\frac{dy}{dx}$ for the implicit relation

$$x^2 y + xy^2 = 6$$

Ans:

$$2x \cancel{\frac{dx}{dx}}^{1} y + x^2 \frac{dy}{dx} + \cancel{\frac{dx}{dx}}^{1} * y^2 + x * 2y \frac{dy}{dx} = 0$$

$$2xy + x^2 \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} = 0$$

$$2xy + y^2 = -x^3 \frac{dy}{dx} - 2xy \frac{dy}{dx}$$

$$(2xy + y^2) = -(x^3 + 2xy)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{-(2xy + y^2)}{(x^2 + 2xy)}$$

Ex: $2xy + y^2 = x + y$

$$2\left[x\frac{dy}{dx} + y\cancelto{1}{\frac{dx}{dx}}\right] + 2y\frac{dy}{dx} = \cancelto{1}{\frac{dx}{dx}} + \frac{dy}{dx}$$

$$2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = \frac{dy}{dx} + 1$$

$$2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$$

$$\therefore \frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$$

$$\frac{dy}{dx} = \frac{1 - 2y}{2x + 2y - 1}$$

Exercises 2.6

Derivatives of Rational Powers

Find dy/dx in Exercises 1–10.

1. $y = x^{9/4}$
2. $y = x^{-3/5}$
3. $y = \sqrt[3]{2x}$
4. $y = \sqrt[4]{5x}$
5. $y = 7\sqrt{x+6}$
6. $y = -2\sqrt{x-1}$
7. $y = (2x+5)^{-1/2}$
8. $y = (1-6x)^{2/3}$
9. $y = x(x^2+1)^{1/2}$
10. $y = x(x^2+1)^{-1/2}$

Find the first derivatives of the functions in Exercises 11–18.

11. $s = \sqrt[7]{t^2}$
12. $r = \sqrt[4]{\theta^{-3}}$
13. $y = \sin\left[(2t+5)^{-2/3}\right]$
14. $z = \cos\left[(1-6t)^{2/3}\right]$
15. $f(x) = \sqrt{1-\sqrt{x}}$
16. $g(x) = 2(2x^{-1/2}+1)^{-1/3}$
17. $h(\theta) = \sqrt[3]{1+\cos(2\theta)}$
18. $k(\theta) = (\sin(\theta+5))^{5/4}$

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 19–32.

19. $x^2y + xy^2 = 6$
20. $x^3 + y^3 = 18xy$
21. $2xy + y^2 = x + y$
22. $x^3 - xy + y^3 = 1$
23. $x^2(x-y)^2 = x^2 - y^2$
24. $(3xy+7)^2 = 6y$
25. $y^2 = \dfrac{x-1}{x+1}$
26. $x^2 = \dfrac{x-y}{x+y}$
27. $x = \tan y$
28. $x = \sin y$
29. $x + \tan(xy) = 0$
30. $x + \sin y = xy$
31. $y\sin\left(\dfrac{1}{y}\right) = 1 - xy$
32. $y^2\cos\left(\dfrac{1}{y}\right) = 2x + 2y$

Find $dr/d\theta$ in Exercises 33–36.

33. $\theta^{1/2} + r^{1/2} = 1$
34. $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$
35. $\sin(r\theta) = \dfrac{1}{2}$
36. $\cos r + \cos\theta = r\theta$

Higher Derivatives

In Exercises 37–42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

37. $x^2 + y^2 = 1$
38. $x^{2/3} + y^{2/3} = 1$
39. $y^2 = x^2 + 2x$
40. $y^2 - 2x = 1 - 2y$
41. $2\sqrt{y} = x - y$
42. $xy + y^2 = 1$
43. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
44. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

Slopes, Tangents, and Normals

In Exercises 45 and 46, find the slope of the curve at the given points.

45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
46. $(x^2+y^2)^2 = (x-y)^2$ at $(1, 0)$ and $(1, -1)$

In Exercises 47–56, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

47. $x^2 + xy - y^2 = 1$, $(2, 3)$
48. $x^2 + y^2 = 25$, $(3, -4)$
49. $x^2y^2 = 9$, $(-1, 3)$
50. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$
51. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$
52. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$
53. $2xy + \pi\sin y = 2\pi$, $(1, \pi/2)$
54. $x\sin 2y = y\cos 2x$, $(\pi/4, \pi/2)$
55. $y = 2\sin(\pi x - y)$, $(1, 0)$
56. $x^2\cos^2 y - \sin y = 0$, $(0, \pi)$
57. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?
58. Find points on the curve $x^2 + xy + y^2 = 7$ (a) where the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis. In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?
59. *The eight curve.* Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.