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عنوان المحاضرة: Trigonometric function

1. Introduction

This unit looks at the solution of trigonometric equations. In order to solve these equations we shall make extensive use of the graphs of the functions sine, cosine and tangent. The symmetries which are apparent in these graphs, and their periodicities are particularly important as we shall see.

2. Some special angles and their trigonometric ratios.

In the examples which follow a number of angles and their trigonometric ratios are used frequently. We list these angles and their sines, cosines and tangents.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	$0°$	$30°$	$45°$	$60°$	$90°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

3. Some simple trigonometric equations

Example

Suppose we wish to solve the equation $\sin x = 0.5$ and we look for all solutions lying in the interval $0° \leq x \leq 360°$. This means we are looking for all the angles, x, in this interval which have a sine of 0.5.

We begin by sketching a graph of the function $\sin x$ over the given interval. This is shown in Figure 1.

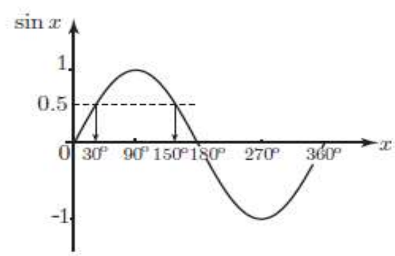

Figure 1. A graph of $\sin x$.

We have drawn a dotted horizontal line on the graph indicating where $\sin x = 0.5$. The solutions of the given equation correspond to the points where this line crosses the curve. From the Table above we note that the first angle with a sine equal to 0.5 is $30°$. This is indicated in Figure 1. Using the symmetries of the graph, we can deduce all the angles which have a sine of 0.5. These are:

$$x = 30°, 150°$$

This is because the second solution, $150°$, is the same distance to the left of $180°$ that the first is to the right of $0°$. There are no more solutions within the given interval.

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Example

Suppose we wish to solve $\sin 2x = \frac{\sqrt{3}}{2}$ for $0 \le x \le 360°$.

Note that in this case we have the sine of a multiple angle, $2x$.

To enable us to cope with the multiple angle we shall consider a new variable u where $u = 2x$, so the problem becomes that of solving

$$\sin u = \frac{\sqrt{3}}{2} \qquad \text{for } 0 \le u \le 720°$$

$$u = 60°, 120°, 420°, 480°$$

But $u = 2x$ so that

$$2x = 60°, 120°, 420°, 480°$$

from which

$$x = 30°, 60°, 210°, 240°$$

Example

Suppose we wish to solve $\cos 2x = \frac{1}{2}$ for $-180° \le x \le 180°$.

In this Example we have a multiple angle, $2x$.

To handle this we let $u = 2x$ and instead solve

$$\cos u = \frac{1}{2} \qquad \text{for } -360° \le x \le 360°$$

$$u = -300°, -60°, 60°, 300°$$

Then $u = 2x$ so that

$$2x = -300°, -60°, 60°, 300°$$

from which

$$x = -150°, -30°, 30°, 150°$$

Example

Suppose we wish to solve $\cos\frac{x}{2} = -\frac{1}{2}$ for values of x in the interval $0 \le x \le 360°$.

In this Example we are dealing with the cosine of a multiple angle, $\frac{x}{2}$.

To enable us to handle this we make a substitution $u = \frac{x}{2}$ so that the equation becomes

$$\cos u = -\frac{1}{2} \qquad \text{for } 0 \le u \le 180°$$

A graph of $\cos u$ over this interval is shown in Figure 5.

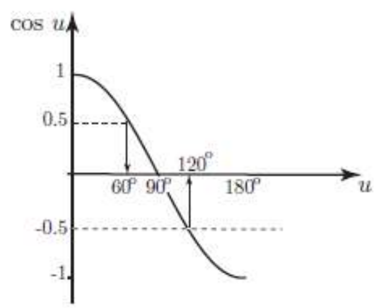

Figure 5. A graph of $\cos u$.

We know that the angle whose cosine is $\frac{1}{2}$ is $60°$. Using the symmetry in the graph we can find all the angles with a cosine equal to $-\frac{1}{2}$. In the interval given there is only one angle with cosine equal to $-\frac{1}{2}$ and that is $u = 120°$

But $u = \frac{x}{2}$ and so $x = 2u$. We conclude that there is a single solution, $x = 240°$.

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Example

Suppose we wish to solve $\tan 2x = \sqrt{3}$ for $-180° \leq x \leq 180°$.

We again have a multiple angle, $2x$. We handle this by letting $u = 2x$ so that the problem becomes that of solving

$$\tan u = \sqrt{3} \qquad \text{for } -360° \leq u \leq 360°$$

$$u = -300°, -120°, 60°, 240°$$

But $u = 2x$ and so

$$2x = -300°, -120°, 60°, 240°$$

and so the required solutions are

$$x = -150°, -60°, 30°, 120°$$

Exercise 1

1. Find all the solutions of each of the following equations in the given range
   (a) $\sin x = \frac{1}{\sqrt{2}}$ for $0 < x < 360^o$
   (b) $\cos x = -\frac{1}{\sqrt{2}}$ for $0 < x < 360^o$
   (c) $\tan x = \frac{1}{\sqrt{3}}$ for $0 < x < 360^o$
   (d) $\cos x = -1$ for $0 < x < 360^o$
2. Find all the solutions of each of the following equations in the given range
   (a) $\tan x = \sqrt{3}$ for $-180^o < x < 180^o$
   (b) $\tan x = -\sqrt{3}$ for $-180^o < x < 180^o$
   (c) $\cos x = \frac{1}{2}$ for $-180^o < x < 180^o$
   (d) $\sin x = -\frac{1}{\sqrt{2}}$ for $-180^o < x < 180^o$

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4. Using identities in the solution of equations

There are many **trigonometric identities**. Two commonly occuring ones are

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x = 1 + \tan^2 x$$

We will now use these in the solution of trigonometric equations. (If necessary you should refer to the unit entitled *Trigonometric Identities*).

Example

Suppose we wish to solve the equation $\cos^2 x + \cos x = \sin^2 x$ for $0° \le x \le 180°$.

We can use the identity $\sin^2 x + \cos^2 x = 1$, rewriting it as $\sin^2 x = 1 - \cos^2 x$ to write the given equation entirely in terms of cosines.

$$\begin{aligned} \cos^2 x + \cos x &= \sin^2 x \\ \cos^2 x + \cos x &= 1 - \cos^2 x \end{aligned}$$

Rearranging, we can write

$$2\cos^2 x + \cos x - 1 = 0$$

This is a quadratic equation in which the variable is $\cos x$. This can be factorised to

$$(2\cos x - 1)(\cos x + 1) = 0$$

Hence

$$2\cos x - 1 = 0 \qquad \text{or} \qquad \cos x + 1 = 0$$

from which

$$\cos x = \frac{1}{2} \qquad \text{or} \qquad \cos x = -1$$

We solve each of these equations in turn. By referring to the graph of $\cos x$ over the interval $0 \le x \le 180°$ which is shown in Figure 9, we see that there is only one solution of the equation $\cos x = \frac{1}{2}$ in this interval, and this is $x = 60°$. From the same graph we can deduce the solution of $\cos x = -1$ to be $x = 180°$.

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Example

Suppose we wish to solve the equation $3\tan^2 x = 2\sec^2 x + 1$ for $0° \le x \le 180°$.

In this example we will simplify the equation using the identity $\sec^2 x = 1 + \tan^2 x$.

$$\begin{aligned} 3\tan^2 x &= 2\sec^2 x + 1 \\ 3\tan^2 x &= 2(1 + \tan^2 x) + 1 \\ 3\tan^2 x &= 2 + 2\tan^2 x + 1 \end{aligned}$$

Rearranging we can write

$$\tan^2 x = 3$$

so that

$$\tan x = +\sqrt{3} \text{ or } -\sqrt{3}$$

We solve each of these equations separately.

The solutions of $\tan x = \sqrt{3}$ can be obtained by inspecting the graph in Figure 10. From the Table on page 2 we know that one angle with a tangent of $\sqrt{3}$ is $60°$. There are no other solutions in the given interval. Using the symmetry of the graph we can deduce the solution of the equation $\tan x = -\sqrt{3}$. This is $x = 120°$.

So the given equation has two solutions, $x = 60°$ and $x = 120°$.

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Example

Suppose we wish to solve the equation $2\cos^2 x + \sin x = 1$ for $0 \le x \le 2\pi$.

We shall use the identity $\sin^2 x + \cos^2 x = 1$ to rewrite the equation entirely in terms of sines.

$$\begin{aligned} 2\cos^2 x + \sin x &= 1 \\ 2(1-\sin^2 x) + \sin x &= 1 \\ 2 - 2\sin^2 x + \sin x &= 1 \end{aligned}$$

and rearranging

$$2\sin^2 x - \sin x - 1 = 0$$

This is a quadratic equation in $\sin x$ which can be factorised to give

$$(2\sin x + 1)(\sin x - 1) = 0$$

Hence

$$2\sin x + 1 = 0 \qquad \text{and} \qquad \sin x - 1 = 0$$

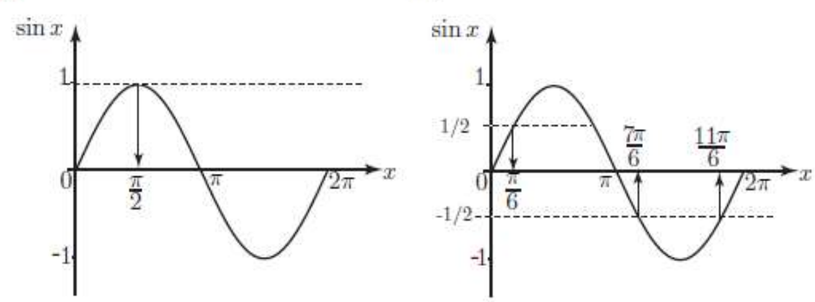
from which

$$\sin x = -\frac{1}{2} \qquad \text{and} \qquad \sin x = 1$$

We solve each of these separately. Graphs of $\sin x$ are shown in Figure 14. From Figure 14(a) we can deduce the solution of $\sin x = 1$ to be $x = \frac{\pi}{2}$. Solutions of $\sin x = -\frac{1}{2}$ can be deduced from Figure 14(b). We know from the Table on page 2 that an angle with sine equal to $\frac{1}{2}$ is 30° or $\frac{\pi}{6}$. Using the symmetry in the graph we can deduce the angles with sine equal to $-\frac{1}{2}$ to be $\pi + \frac{\pi}{6}$ and $2\pi - \frac{\pi}{6}$. Hence

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

(a) (b)



So, the full set of solutions of the given equation is $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

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