

5-5- Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27) \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28) \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29) \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u| < 1 \\ \coth^{-1} u + c & \text{if } |u| > 1 \end{cases} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + c$$

$$30) \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$$

$$31) \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$$

EX-4 – Evaluate the following integrals:

$$1) \int \frac{dx}{\sqrt{1+4x^2}}$$

$$2) \int \frac{dx}{\sqrt{4+x^2}}$$

$$3) \int \frac{dx}{1-x^2}$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}}$$

$$5) \int \frac{\sec^2 \theta d\theta}{\sqrt{\tan^2 \theta - 1}}$$

$$6) \int \tanh^{-1}(\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})}$$

Sol.-

$$1) \frac{1}{2} \int \frac{2 dx}{\sqrt{1+4x^2}} = \frac{1}{2} \sinh^{-1} 2x + c$$

$$2) \int \frac{\frac{1}{2} dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1} \frac{x}{2} + c$$

$$3) \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad \text{if } |x| < 1 \\ = \coth^{-1} x + c \quad \text{if } |x| > 1$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2} \int \frac{\frac{1}{2} dx}{\frac{x}{2} \sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2} \operatorname{csc} h^{-1} \left| \frac{x}{2} \right| + c$$

$$5) \int \frac{1}{\sqrt{\tan^2 \theta - 1}} (\sec^2 \theta d\theta) = \operatorname{cosh}^{-1}(\tan \theta) + c$$

$$6) \quad \text{let } u = \ln \sqrt{x} = \frac{1}{2} \ln x \quad du = \frac{1}{2x} dx$$

$$\begin{aligned} \int \tanh^{-1}(\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})} &= \int \tanh^{-1} u \cdot \frac{2 du}{1-u^2} \\ &= 2 \frac{(\tanh^{-1} u)^2}{2} + c = \left[\tanh^{-1}(\ln \sqrt{x}) \right]^2 + c \end{aligned}$$

Problems – 5

Evaluate the following integrals:

- 1) $\int (x^2 - 1) \cdot (4 - x^2) dx$ (ans.: $\frac{5}{3}x^3 - \frac{1}{5}x^5 - 4x + c$)
- 2) $\int e^x \cdot \sin e^x dx$ (ans.: $-\cos e^x + c$)
- 3) $\int \tan(3x + 5) dx$ (ans.: $-\frac{1}{3} \ln |\cos(3x + 5)| + c$)
- 4) $\int \frac{\cot(\ln x)}{x} dx$ (ans.: $\ln |\sin(\ln x)| + c$)
- 5) $\int \frac{\sin x + \cos x}{\cos x} dx$ (ans.: $-\ln |\cos x| + x + c$)
- 6) $\int \frac{dx}{1 + \cos x}$ (ans.: $-\cot x + \csc x + c$)
- 7) $\int \cot(2x + 1) \cdot \csc^2(2x + 1) dx$ (ans.: $-\frac{1}{4} \cot^2(2x + 1) + c$)
- 8) $\int \frac{dx}{\sqrt{1 - 9x^2}}$ (ans.: $\frac{1}{3} \sin^{-1}(3x) + c$)
- 9) $\int \frac{dx}{\sqrt{2 - x^2}}$ (ans.: $\sin^{-1} \frac{x}{\sqrt{2}} + c$)
- 10) $\int e^{2x} \cdot \cos e^{2x} dx$ (ans.: $\frac{1}{2} \sinh e^{2x} + c$)
- 11) $\int e^{\sin x} \cdot \cos x dx$ (ans.: $e^{\sin x} + c$)
- 12) $\int \frac{dx}{e^{3x}}$ (ans.: $-\frac{1}{3} e^{-3x} + c$)
- 13) $\int \frac{e^{\sqrt{x}} - 1}{\sqrt{x}} dx$ (ans.: $2e^{\sqrt{x}} - 2\sqrt{x} + c$)
- 14) $\int x(a + b\sqrt{3x}) dx$ where a, b constants (ans.: $\frac{1}{10}(5ax^2 + 4\sqrt{3}bx^{5/2}) + c$)
- 15) $\int \frac{dx}{-1 - x^2}$ (ans.: $-\tan^{-1} x + c$)
- 16) $\int \frac{\cos \theta d\theta}{1 + \sin^2 \theta}$ (ans.: $\tan^{-1}(\sin \theta) + c$)

- 17) $\int \frac{1}{x^2} \csc \frac{1}{x} \cot \frac{1}{x} dx$ (ans.: $\csc \frac{1}{x} + c$)
- 18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}} dx$ (ans.: $\frac{3}{4} \sqrt[3]{(3x^2+2x+1)^2} + c$)
- 19) $\int \sin(\tan \theta) \cdot \sec^2 \theta d\theta$ (ans.: $-\cos(\tan \theta) + c$)
- 20) $\int \sqrt{x^2 - x^4} dx$ (ans.: $-\frac{1}{3} \sqrt{(1-x^2)^3} + c$)
- 21) $\int \frac{\sec^2 2x dx}{\sqrt{\tan 2x}}$ (ans.: $\sqrt{\tan 2x} + c$)
- 22) $\int (\sin \theta - \cos \theta)^2 d\theta$ (ans.: $\theta + \cos^2 \theta + c$)
- 23) $\int \frac{y}{y^4 + 1} dy$ (ans.: $\frac{1}{2} \tan^{-1} y^2 + c$)
- 24) $\int \frac{dx}{\sqrt{x(x+1)}}$ (ans.: $2 \tan^{-1} \sqrt{x} + c$)
- 25) $\int t^{\frac{2}{3}} (t^{\frac{5}{3}} + 1)^{\frac{2}{3}} dt$ (ans.: $\frac{9}{25} (t^{\frac{5}{3}} + 1)^{\frac{5}{3}} + c$)
- 26) $\int \frac{dx}{x^{\frac{1}{5}} \sqrt{1+x^{\frac{4}{5}}}}$ (ans.: $\frac{5}{2} \sqrt{1+x^{\frac{4}{5}}} + c$)
- 27) $\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}} dx$ (ans.: $-\frac{1}{12} (\cos^{-1} 4x)^3 + c$)
- 28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ (ans.: $\sec^{-1}(2x) + c$)
- 29) $\int \frac{dx}{(e^x + e^{-x})^2}$ (ans.: $\frac{1}{4} \tanh x + c$)
- 30) $\int 3^{\ln x^2} \frac{dx}{x}$ (ans.: $\frac{1}{2 \ln 3} 3^{\ln x^2} + c$)
- 31) $\int \frac{\cot x dx}{\ln(\sin x)}$ (ans.: $\ln \ln(\sin x) + c$)
- 32) $\int \frac{(\ln x)^2}{x} dx$ (ans.: $\frac{1}{3} (\ln x)^3 + c$)
- 33) $\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x} dx$ (ans.: $e^{\sec x} + c$)

- 34) $\int \frac{dx}{x \cdot \ln x}$ (ans.: $\ln \ln x + c$)
- 35) $\int \frac{d\theta}{\cosh \theta + \sinh \theta}$ (ans.: $-e^{-\theta} + c$)
- 36) $\int \frac{2^x - 8^{2x}}{\sqrt{4^x}} dx$ (ans.: $x - \frac{1}{5 \ln 2} 2^{5x} + c$)
- 37) $\int \frac{e^{\tan^{-1} 2t}}{1+4t^2} dt$ (ans.: $\frac{1}{2} e^{\tan^{-1} 2t} + c$)
- 38) $\int \frac{\cot x}{\csc x} dx$ (ans.: $\sin x + c$)
- 39) $\int \sec^4 x \cdot \tan^3 x dx$ (ans.: $\frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + c$)
- 40) $\int \csc^4 3x dx$ (ans.: $-\frac{1}{9} \cot^3 3x - \frac{1}{3} \cot 3x + c$)
- 41) $\int \frac{\cos^3 t}{\sin^2 t} dt$ (ans.: $-\operatorname{csc} t - \sin t + c$)
- 42) $\int \frac{\sec^4 x}{\tan^4 x} dx$ (ans.: $-\frac{1}{3} \cot^3 x - \cot x + c$)
- 43) $\int \tan^2 4\theta d\theta$ (ans.: $\frac{1}{4} \tan 4\theta - \theta + c$)
- 44) $\int \frac{e^x}{1+e^x} dx$ (ans.: $\ln(1+e^x) + c$)
- 45) $\int \tan^3 2x dx$ (ans.: $\frac{1}{4} \tan^2 2x + \frac{1}{2} \ln|\cos 2x| + c$)
- 46) $\int \frac{\sec^2 x}{2 + \tan x} dx$ (ans.: $\ln(2 + \tan x) + c$)
- 47) $\int \sec^4 3x dx$ (ans.: $\frac{1}{9} \tan^3 3x + \frac{1}{3} \tan 3x + c$)
- 48) $\int \frac{e^t}{1+e^{2t}} dt$ (ans.: $\tan^{-1} e^t + c$)
- 49) $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ (ans.: $2 \sin \sqrt{x} + c$)
- 50) $\int \frac{dx}{\sin x \cdot \cos x}$ (ans.: $-\ln|\csc 2x + \cot 2x| + c$)

- 51) $\int \sqrt{1 + \sin y} \, dy$ (ans.: $-2\sqrt{1 - \sin y} + c$)
- 52) $\int \frac{dx}{(x^2 + 1)(2 + \tan^{-1} x)}$ (ans.: $\ln(2 + \tan^{-1} x) + c$)
- 53) $\int \sin^{-1}(\cosh x) \cdot \frac{\sinh x \, dx}{\sqrt{1 - \cosh^2 x}}$ (ans.: $\frac{1}{2}(\sinh^{-1}(\cosh x))^2 + c$)
- 54) $\int \frac{\cos \theta \, d\theta}{1 - \sin^2 \theta}$ (ans.: $\ln|\sec \theta + \tan \theta| + c$)
- 55) $\int \frac{dx}{x(1 + (\ln x)^2)}$ (ans.: $\tan^{-1}(\ln x) + c$)
- 56) $\int \left(e^{\frac{2}{4}x} - 2e^{\frac{5}{4}x} + e^{\frac{x}{4}} \right) dx$ (ans.: $\frac{4}{9}e^{\frac{2}{4}x} - \frac{8}{5}e^{\frac{5}{4}x} + 4e^{\frac{x}{4}} + c$)
- 57) $\int \frac{e^x \, dx}{e^{2x} + 2e^x + 1}$ (ans.: $-\frac{1}{e^x + 1} + c$)
- 58) $\int e^x \cdot \sinh 2x \, dx$ (ans.: $\frac{1}{2} \left[\frac{1}{3}e^{3x} + e^{-x} \right] + c$)
- 59) $\int \frac{\sec^3 x + e^{\sin x}}{\sec x} \, dx$ (ans.: $\tan x + e^{\sin x} + c$)
- 60) $\int \frac{3^{x+2}}{2 + 9^{x+1}} \, dx$ (ans.: $\frac{3}{\sqrt{2} \ln 3} \tan^{-1} \frac{3^{x+1}}{\sqrt{2}} + c$)
- 61) $\int \frac{\cos x \, dx}{\sqrt{\sin x} \cdot \sqrt{1 - \sin x}}$ (ans.: $2\sin^{-1} \sqrt{\sin x} + c$)
- 62) $\int \tan^5 x \, dx$ (ans.: $\frac{1}{4} \sec^4 x - \sec^2 x - \ln|\cos x| + c$)
- 63) $\int e^{\ln \sin^{-1} x} \cdot \frac{dx}{\sqrt{1 - x^2}}$ (ans.: $\frac{1}{2}(\sin^{-1} x)^2 + c$)
- 64) $\int x e^{x^2-1} \, dx$ (ans.: $\frac{1}{2}e^{x^2-1} + c$)
- 65) $\int \cosh(\ln \cos x) \, dx$ (ans.: $\frac{1}{2}[\sin x + \ln|\sec x + \tan x|] + c$)
- 66) $\int \frac{\cos x}{\sin^2 x} \, dx$ (ans.: $-\csc x + c$)
- 67) $\int \cosh^{-1}(\sin x) \frac{\cos x \, dx}{\sqrt{\sin^2 x - 1}}$ (ans.: $\frac{1}{2}[\cosh^{-1}(\sin x)]^2 + c$)