Chapter six

Methods of integration

6-1- <u>Integration by parts</u>:

The formula for integration by parts comes from the product rule:-

$$d(u \cdot v) = u \cdot dv + v \cdot du \implies u \cdot dv = d(u \cdot v) - v \cdot du$$

and integrated to give: $\int u \, dv = \int d(u \cdot v) - \int v \, du$

then the integration by parts formula is:-

$$\int u \, dv = u \cdot v - \int v \, du$$

Rule for choosing u and dv is:

For **u**: choose something that becomes simpler when differentiated.

For **dv**: choose something whose integral is simple.

It is not always possible to follow this rule, but when we can.

<u>EX-1</u> – Evaluate the following integrals:

1) $\int xe^x \, dx$

2) $\int x \cdot cosx \, dx$

3) $\int \dfrac{x}{\sqrt{x-1}} \, dx$

4) $\int x^2 \cdot \ln x \, dx$

5) $\int x \cdot sec^2 x \, dx$

6) $\int \ln\left(x + \sqrt{1+x^2}\right) dx$

7) $\int sin^{-1}ax \, dx$

8) $\int e^{ax} \cdot sinbx \, dx$

9) $\int x^3 \cdot e^x \, dx$

10) $\int x^3 \cdot e^{x^2} \, dx$

<u>Sol.</u> –

1) let $\left. \begin{array}{l} u = x \implies du = dx \\ dv = e^x dx \implies v = e^x \end{array} \right\} \implies \int u dv = u \cdot v - \int v du$

$$\int x \cdot e^x dx = x \cdot e^x - \int e^x dx = x \cdot e^x - e^x + c$$

2) let $\left.\begin{array}{l}u = x \implies du = dx \\ dv = cosx\ dx \implies v = sin\ x\end{array}\right\} \implies \int u\,dv = u \cdot v - \int v\,du$

$$\int x \cdot cosx\ dx = x \cdot sin\ x - \int sin\ x\ dx = x \cdot sin\ x + cos\ x + c$$

3) let $\left.\begin{array}{l}u = x \implies du = dx \\ dv = \dfrac{1}{\sqrt{x-1}}dx \implies v = 2(x-1)^{1/2}\end{array}\right\} \implies \int u\,dv = u \cdot v - \int v\,du$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x \cdot (x-1)^{1/2} - 2\int (x-1)^{1/2}dx$$

$$= 2x \cdot \sqrt{x-1} - \frac{2(x-1)^{3/2}}{3/2} + c = 2x \cdot \sqrt{x-1} - \frac{4}{3}\sqrt{(x-1)^3} + c$$

4) let $\left.\begin{array}{l}u = ln\ x \implies du = \dfrac{1}{x}dx \\ dv = x^2 dx \implies v = \dfrac{x^3}{3}\end{array}\right\} \implies \int u\,dv = u \cdot v - \int v\,du$

$$\int x^2 \cdot lnx\ dx = \frac{x^3}{3} \cdot ln\ x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3} \cdot ln\ x - \frac{1}{9}x^3 + c$$

5) let $\left.\begin{array}{l}u = x \implies du = dx \\ dv = sec^2x\ dx \implies v = tan\ x\end{array}\right\} \implies \int u\,dv = u \cdot v - \int v\,du$

$$\int x \cdot sec^2\ x\ dx = x \cdot tan\ x - \int tan\ x\ dx = x \cdot tan\ x + ln|cos\ x| + c$$

6) let $u = ln\left(x + \sqrt{1+x^2}\right) \implies du = \dfrac{1 + \dfrac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}dx$

$dv = dx \implies v = x$

$$\int ln\left(x + \sqrt{1+x^2}\right)dx = x \cdot ln\left(x + \sqrt{1+x^2}\right) - \int x\left(1+x^2\right)^{-1/2}dx$$

$$= x \cdot ln\left(x + \sqrt{1+x^2}\right) - \frac{1}{2} \cdot \frac{\left(1+x^2\right)^{1/2}}{1/2} + c = x \cdot ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + c$$

7) let $u = \sin^{-1} ax \Rightarrow du = \dfrac{a\,dx}{\sqrt{1-a^2x^2}}$ & $dv = dx \Rightarrow v = x$

$$\int \sin^{-1} ax\,dx = x \cdot \sin^{-1} ax - \int \frac{a\,x}{\sqrt{1-a^2x^2}}\,dx$$

$$= x \cdot \sin^{-1} ax + \frac{1}{2a}\int -2a^2 x\left(1-a^2x^2\right)^{-\frac{1}{2}} dx$$

$$= x \cdot \sin^{-1} ax + \frac{1}{2a} \cdot \frac{\left(1-a^2x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \cdot \sin^{-1} ax + \frac{\sqrt{1-a^2x^2}}{a} + c$$

8) let $u = e^{ax} \Rightarrow du = a \cdot e^{ax}\,dx$ & $dv = \sin bx\,dx \Rightarrow v = -\dfrac{1}{b}\cos bx$

$$\int e^{ax} \cdot \sin bx\,dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b}\int e^{ax} \cdot \cos bx\,dx \quad ....(1)$$

let $u = e^{ax} \Rightarrow du = a \cdot e^{ax}\,dx$ & $dv = \cos bx\,dx \Rightarrow v = \dfrac{1}{b}\sin bx$

$$\int e^{ax} \cdot \cos bx\,dx = \frac{1}{b} e^{ax} \cdot \sin bx - \frac{a}{b}\int e^{ax} \cdot \sin bx\,dx \quad ....(2)$$

sub. (2) in (1) $\Rightarrow$

$$\int e^{ax} \cdot \sin bx\,dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx\,dx - \frac{a^2}{b^2}\int e^{ax} \cdot \sin bx\,dx$$

$$\int e^{ax} \cdot \sin bx\,dx + \frac{a^2}{b^2}\int e^{ax} \cdot \sin bx\,dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx\,dx + c$$

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax} \cdot \sin bx\,dx = \frac{e^{ax}}{b^2}\left(a \sin bx - b \cos bx\right) + c$$

$$\therefore \quad \int e^{ax} \cdot \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a \sin bx - b \cos bx\right) + c$$

9) <u>*derivative of u*</u>   <u>*integration of dv*</u>

x^3	$+$	e^x
$3x^2$	$-$	e^x
$6x$	$+$	e^x
6	$-$	e^x
0		e^x

$$\therefore \int x^3 e^{ax}\,dx = x^3 e^x - 3x^2 e^x$$

$$+ 6xe^x - 6e^x + c$$

$$= e^x\left(x^3 - 3x^2 + 6x - 6\right) + c$$

10)   let $u = x^2$ $\Rightarrow$ $du = 2x\,dx$ & $dv = x \cdot e^{x^2} dx \Rightarrow v = \dfrac{1}{2} e^{x^2}$

$$\int x^3 \cdot e^{x^2} dx = \frac{1}{2} x^2 \cdot e^{x^2} - \frac{1}{2} \int 2x \cdot e^{x^2} dx = \frac{1}{2} x^2 \cdot e^{x^2} - \frac{1}{2} e^{x^2} + c$$

6-2- <u>Odd and even powers of sine and cosine:</u>

      To integrate an odd positive power of $sinx$ (say $sin^{2n+1}x$) we split off a factor of $sinx$ and rewrite the remaining even power in terms of the cosine. We write:-

$$\int sin^{2n+1} x \cdot dx = \int \left(1 - cos^2 x\right)^n \cdot sin x \; dx$$

and $\int cos^{2n+1} x \cdot dx = \int \left(1 - sin^2 x\right)^n \cdot cos x \; dx$

   <u>*EX-2*</u>- **Evaluate:**

    1) $\int sin^3 x \, dx$ **2)** $\int cos^5 x \, dx$

<u>Sol.</u>-

1) $\int sin^3 x \, dx = \int sin^2 x \cdot sinx \, dx = \int \left(1 - cos^2 x\right) \cdot sinx \, dx$

$$= \int sinx \, dx + \int cos^2 x \cdot \left(- sinx\right) dx = - cos \, x + \frac{1}{3} cos^3 x + c$$

2) $\int cos^5 x \, dx = \int cos^4 x \cdot cosx \, dx = \int \left(1 - sin^2 x\right)^2 \cdot cos x \, dx$

$$= \int cosx \, dx - 2 \int sin^2 x \cdot cos x \, dx + \int sin^4 x \cdot cos x \, dx$$

$$= sinx - \frac{2}{3} sin^3 x + \frac{1}{5} sin^5 x + c$$

      To integrate an even positive power of sine (say $sin^{2n}x$) we use the relations:-

$$cos^2 \theta = \frac{1 + cos 2\theta}{2} \qquad or \qquad sin^2 \theta = \frac{1 - cos 2\theta}{2}$$

then we can write:-

$$\int \sin^{2n} x \cdot dx = \int \left(\frac{1 - \cos 2x}{2} \right)^n dx$$

$$\text{and } \int \cos^{2n} x \cdot dx = \int \left(\frac{1 + \cos 2x}{2} \right)^n dx$$

EX-3- **Evaluate:**

1) $\int \cos^2 \theta \, d\theta$ 2) $\int \sin^4 \theta \, d\theta$

Sol.-

1) $\int \cos^2\theta \, d\theta = \int \dfrac{1 + \cos 2\theta}{2} d\theta = \dfrac{1}{2}\left[\int d\theta + \dfrac{1}{2}\int 2\cos 2\theta \, d\theta \right]$

$$= \dfrac{1}{2}\left[\theta + \dfrac{1}{2} \sin 2\theta \right] + c$$

2) $\int \sin^4\theta \, d\theta = \int \left(\dfrac{1 - \cos 2\theta}{2} \right)^2 d\theta = \dfrac{1}{4}\left[\int d\theta - \int \cos 2\theta (2d\theta) + \int \cos^2 2\theta \, d\theta \right]$

$$= \dfrac{1}{4}\left[\theta - \sin 2\theta + \int \dfrac{1 + \cos 4\theta}{2} d\theta \right] = \dfrac{1}{4}\left[\theta - \sin 2\theta + \dfrac{1}{2}(\theta + \dfrac{1}{4} \sin 4\theta) \right] + c$$

$$= \dfrac{3}{8}\theta - \dfrac{1}{4} \sin 2\theta + \dfrac{1}{32} \sin 4\theta + c$$

To integrate the following identities:-

$$\int \sin mx \cdot \sin nx \, dx \ , \ \int \sin mx \cdot \cos nx \, dx \ , and \ \int \cos mx \cdot \cos nx \, dx$$

we use the following formulas:-

$$\sin mx \cdot \sin nx = \dfrac{\cos(m - n)x - \cos(m + n)x}{2}$$

$$\sin mx \cdot \cos nx = \dfrac{\sin(m - n)x + \sin(m + n)x}{2}$$

$$\cos mx \cdot \cos nx = \dfrac{\cos(m - n)x + \cos(m + n)x}{2}$$

<u>*EX-4*</u>- **Evaluate:**

$$1)\ \int sin\,3x\cdot cos\,5x\,dx \qquad 2)\ \int cos\,x\cdot cos\,7x\,dx \qquad 3)\int sin\,x\cdot sin\,2x\,dx$$

<u>*Sol.*-</u>

$$1)\ \int sin\,3x\cdot cos\,5x\,dx = \frac{1}{2}\int\big(sin(\,3x-5x\,)+sin(\,3x+5x\,)\big)\,dx$$

$$=\frac{1}{2}\left[-\frac{1}{2}\int sin\,2x(\,2dx\,)+\frac{1}{8}\int sin\,8x(\,8dx\,)\right]=\frac{1}{4}cos\,2x-\frac{1}{16}cos\,8x+c$$

$$2)\ \int cos\,x\cdot cos\,7x\,dx = \frac{1}{2}\int\big(cos(\,6x\,)+cos(\,8x\,)\big)\,dx = \frac{1}{12}sin\,6x+\frac{1}{16}sin\,8x+c$$

$$3)\ \int sin\,x\cdot sin\,2x\,dx = \frac{1}{2}\int\big(cos\,x-cos\,3x\big)\,dx = \frac{1}{2}sin\,x-\frac{1}{6}sin\,3x+c$$

6-3- <u>Trigonometric substitutions</u>:

Trigonometric substitutions enable us to replace the binomials a^2-u^2 , a^2+u^2 , *and* u^2-a^2 be single square terms. We can use:-

$$u = a\,sin\theta \quad for \quad a^2-u^2 = a^2-a^2\,sin^2\,\theta = a^2(\,1-sin^2\,\theta\,)= a^2\,cos^2\,\theta$$

$$u = a\,tan\theta \quad for \quad a^2+u^2 = a^2+a^2\,tan^2\,\theta = a^2(\,1+tan^2\,\theta\,)= a^2\,sec^2\,\theta$$

$$u = a\,sec\theta \quad for \quad u^2-a^2 = a^2\,sec^2\,\theta-a^2 = a^2(sec^2\,\theta-1)= a^2\,tan^2\,\theta$$

<u>*EX-5*</u> **Evaluate the following integrals:**

$$1)\ \int\frac{z^5\,dz}{\sqrt{1+z^2}} \qquad\qquad 4)\ \int\frac{x^2}{\sqrt{9-x^2}}\,dx$$

$$2)\ \int\frac{dx}{\sqrt{4+x^2}} \qquad\qquad 5)\ \int\frac{dt}{\sqrt{25t^2-9}}$$

$$3)\ \int\frac{dx}{4-x^2} \qquad\qquad 6)\ \int\frac{dy}{\sqrt{25+9y^2}}$$

<u>*Sol.-*</u>

1)   *let* $z = \tan\theta$ $\Rightarrow$ $dz = \sec^2\theta \cdot d\theta$ $\tan\theta = \dfrac{z}{1}$

$$\int \frac{z^5\,dz}{\sqrt{1+z^2}} = \int \frac{\tan^5\theta \cdot \sec^2\theta\;d\theta}{\sqrt{1+\tan^2\theta}} = \int \tan^5\theta \cdot \sec\theta\;d\theta$$

$$= \int \tan\theta \cdot \sec\theta \left(\sec^2\theta - 1\right)^2 d\theta$$

$$= \int \sec^4\theta\left(\tan\theta \cdot \sec\theta\,d\theta\right) - 2\int \sec^2\theta\left(\tan\theta \cdot \sec\theta\,d\theta\right) + \int \tan\theta \cdot \sec\theta\,d\theta$$

$$= \frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta + c$$

$$= \frac{1}{5}\left(\sqrt{1+z^2}\right)^5 - \frac{2}{3}\left(\sqrt{1+z^2}\right)^3 + \sqrt{1+z^2} + c$$
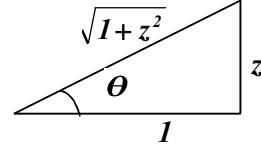
2)   *let* $x = 2\tan\theta$ $\Rightarrow$ $dx = 2\sec^2\theta \cdot d\theta$ $\tan\theta = \dfrac{x}{2}$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\;d\theta}{\sqrt{4+4\tan^2\theta}} = \int \sec\theta\;d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

$$= \ln\left|\sqrt{4+x^2} + x\right| + c' \qquad where\;\; c' = c - \ln 2$$
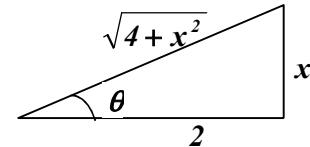
*3)*   *let* $x = 2\sin\theta$ $\Rightarrow$ $dx = 2\cos\theta \cdot d\theta$

$$\int \frac{dx}{4-x^2} = \int \frac{2\cos\theta\;d\theta}{4-4\sin^2\theta} = \frac{1}{2}\int \frac{d\theta}{\cos\theta} = \frac{1}{2}\int \sec\theta\;d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + c$$

$$= \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{(2-x)(2+x)}}\right| + c = \frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right| + c = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + c$$
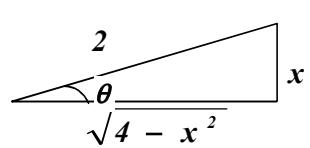
**4)** let $x = 3\sin\theta \implies dx = 3\cos\theta \cdot d\theta$

$$\int \frac{x^2\, dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\, 3\cos\theta\, d\theta = 9\int \sin^2\theta\, d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$

$$= \frac{9}{2}\left(\theta - \sin\theta \cdot \cos\theta\right) + c$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + c = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\cdot\sqrt{9-x^2} + c$$

triangle with hypotenuse 3, opposite x, adjacent $\sqrt{9-x^2}$, angle θ
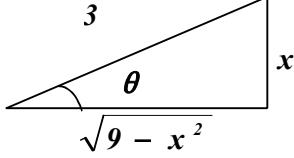
5) let $5t = 3\sec\theta \implies 5dt = 3\sec\theta \cdot \tan\theta\, d\theta$

$$\int \frac{dt}{\sqrt{25t^2-9}} = \int \frac{{}^{3}\!\!\diagup\!\!{}_{5}\,\sec\theta\cdot\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}} = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + c$$

$$= \frac{1}{5}\ln\left|\frac{5t}{3} + \frac{\sqrt{25t^2-9}}{3}\right| + c$$

$$= \frac{1}{5}\ln\left|5t + \sqrt{25t^2-9}\right| + c' \quad where \ \ c' = c - \frac{1}{5}\ln 3$$

triangle with hypotenuse $5t$, opposite $\sqrt{25t^2-9}$, adjacent 3, angle θ
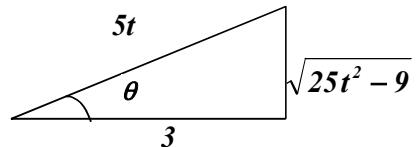
6) let $3y = 5\tan\theta \implies 3dy = 5\sec^2\theta\, d\theta$

$$\int \frac{dy}{\sqrt{25+9y^2}} = \int \frac{{}^{5}\!\!\diagup\!\!{}_{3}\,\sec^2\theta\, d\theta}{\sqrt{25+25\tan^2\theta}} = \frac{1}{3}\int \sec\theta\, d\theta$$

$$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + c$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5}\right| + c$$

$$= \frac{1}{3}\ln\left|\sqrt{25+9y^2} + 3y\right| + c' \quad where \ \ c' = c - \frac{1}{3}\ln 5$$

triangle with hypotenuse $\sqrt{25+9y^2}$, opposite $3y$, adjacent 5, angle θ
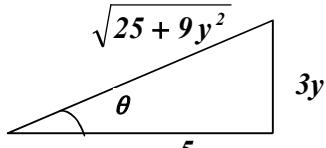
<u>**EX-6**</u> **Prove the following formulas:**

$$1)\ \int \frac{du}{\sqrt{a^2 - u^2}} = sin^{-1} \frac{u}{a} + c \qquad\qquad 2)\ \int \frac{du}{a^2 + u^2} = \frac{1}{a} tan^{-1} \frac{u}{a} + c$$

<u>*Proof.-*</u>

$1)$ *let* $u = a\ sin\ \theta \quad \Rightarrow \quad du = a\ cos\theta \cdot d\theta$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a\ cos\theta \cdot d\theta}{\sqrt{a^2 - a^2\ sin^2\ \theta}} = \int d\theta = \theta + c = sin^{-1} \frac{u}{a} + c$$

$2)$ *let* $u = a\ tan\ \theta \quad \Rightarrow \quad du = a\ sec^2\theta \cdot d\theta$

$$\int \frac{du}{a^2 + u^2} = \int \frac{a\ sec^2\theta \cdot d\theta}{a^2 + a^2\ tan^2\ \theta} = \frac{1}{a} \int d\theta = \frac{1}{a} \theta + c = \frac{1}{a} tan^{-1} \frac{u}{a} + c$$

6-4- <u>**Integral involving $a\,x^2 + b\,x + c$:**</u>

By using the algebraic process called completing the square, we can convert any quadratic: $a\,x^2 + b\,x + c$, $a \neq 0$ to the form: $a(u^2 \mp A^2)$ we can then use one of the trigonometric substitutions to write the expression as a times a single square term.

<u>**EX-7**</u> **– Evaluate:**

$$1)\ \int \frac{dx}{\sqrt{2x - x^2}} \qquad\qquad 4)\ \int \frac{dx}{\sqrt{1 + x - x^2}}$$

$$2)\ \int \frac{dx}{2x^2 + 2x + 1} \qquad\qquad 5)\ \int \frac{dx}{\sqrt{x^2 - 2x - 8}}$$

$$3)\ \int \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

<u>*Sol.*</u>

$$1)\ \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1 - (x^2 - 2x + 1)}} = \int \frac{dx}{\sqrt{1 - (x - 1)^2}}$$

$\quad$ *let* $x - 1 = sin\ \theta \quad \Rightarrow \quad dx = cos\theta\ d\theta$

$$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{cos\theta\ d\theta}{\sqrt{1 - sin^2\ \theta}} = \int d\theta = \theta + c = sin^{-1}(x - 1) + c$$

2) $\displaystyle\int\frac{dx}{2x^{2}+2x+1}=\frac{1}{2}\int\frac{dx}{x^{2}+x+\frac{1}{2}}=\frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^{2}+\frac{1}{4}}$

let $\quad x+\dfrac{1}{2}=\dfrac{1}{2}\tan\theta\quad\Rightarrow\quad dx=\dfrac{1}{2}\sec^{2}\theta\,d\theta$

$\displaystyle\int\frac{dx}{2x^{2}+2x+1}=\frac{1}{2}\int\frac{\frac{1}{2}\sec^{2}\theta\,d\theta}{\frac{1}{4}\tan^{2}\theta+\frac{1}{4}}=\int d\theta=\theta+c=\tan^{-1}(2x+1)+c$

3) $\displaystyle\int\frac{dx}{\sqrt{x^{2}+2x+2}}=\int\frac{dx}{\sqrt{(x+1)^{2}+1}}$

let $\quad x+1=\tan\theta\quad\Rightarrow\quad dx=\sec^{2}\theta\,d\theta$
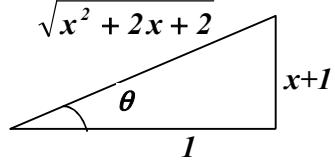
$\displaystyle\int\frac{dx}{\sqrt{x^{2}+2x+2}}=\int\frac{\sec^{2}\theta\,d\theta}{\sqrt{\tan^{2}\theta+1}}=\int\sec\theta\,d\theta$

$=\ln|\sec\theta+\tan\theta|+c=\ln\left|\sqrt{x^{2}+2x+2}+x+1\right|+c$

4) $\displaystyle\int\frac{dx}{\sqrt{1+x-x^{2}}}=\int\frac{dx}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^{2}}}$

let $\quad x-\dfrac{1}{2}=\dfrac{\sqrt{5}}{2}\sin\theta\quad\Rightarrow\quad dx=\dfrac{\sqrt{5}}{2}\cos\theta\,d\theta$

$=\displaystyle\int\frac{\frac{\sqrt{5}}{2}\cos\theta\,d\theta}{\sqrt{\frac{5}{4}-\frac{5}{4}\sin^{2}\theta}}=\int d\theta=\theta+c=\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$

5) $\displaystyle\int\frac{dx}{\sqrt{x^{2}-2x-8}}=\int\frac{dx}{\sqrt{(x-1)^{2}-9}}$

let $\quad x-1=3\sec\theta\quad\Rightarrow\quad dx=3\sec\theta\cdot\tan\theta\,d\theta$

$=\displaystyle\int\frac{3\sec\theta\cdot\tan\theta\,d\theta}{\sqrt{9\sec^{2}\theta-9}}=\int\sec\theta\,d\theta$
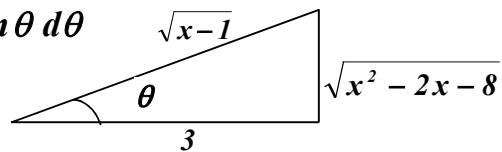
$=\ln|\sec\theta+\tan\theta|+c=\ln\left|\dfrac{x-1}{3}+\dfrac{\sqrt{x^{2}-2x-8}}{3}\right|+c$

$=\ln\left|x-1+\sqrt{x^{2}-2x-8}\right|+c'\qquad where\quad c'=c-\ln 3$