## 4-4- Maxima and Minima :

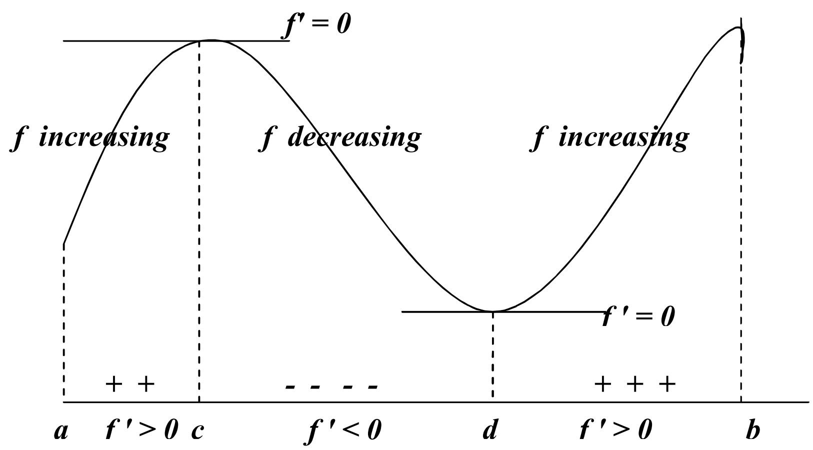
***Increasing and decreasing function :*** Let $f$ be defined on an interval and $x_1 , x_2$ denoted a number on that interval :

- If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then $f$ is increasing on that interval .
- If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then $f$ is decreasing on that interval .
- If $f(x_1) = f(x_2)$ for all values of $x_1 , x_2$ then $f$ is constant on that interval .

***The first derivative test for rise and fall :*** Suppose that a function $f$ has a derivative at every point $x$ of an interval $I$. Then :

- $f$ increases on $I$ if $f'(x) > o, \quad \forall x \in I$
- $f$ decreases on $I$ if $f'(x) > o, \quad \forall x \in I$

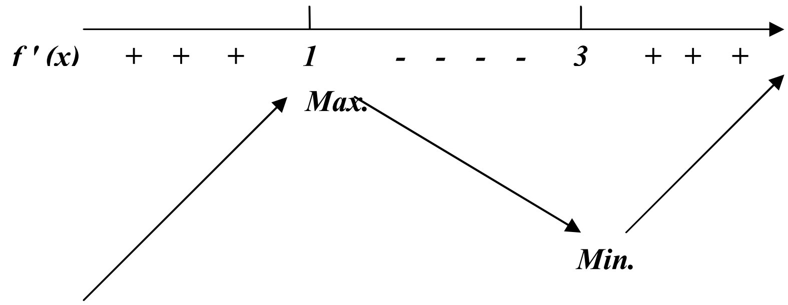
If $f'$ changes from positive to negative values as $x$ passes from left to right through a point $c$ , then the value of $f$ at $c$ is a local maximum value of $f$ , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$ .




Similarly , if $f'$ changes from negative to positive values as $x$ passes left to right through a point $d$ , then the value of $f$ at $d$ is a local minimum value of $f$ . That is $f(d)$ is the smallest value of $f$ takes in the immediate neighborhood of $d$ .

***EX-5*** – Graph the function : $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2$ .

***Sol.***- $f'(x) = x^2 - 4x + 3 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$

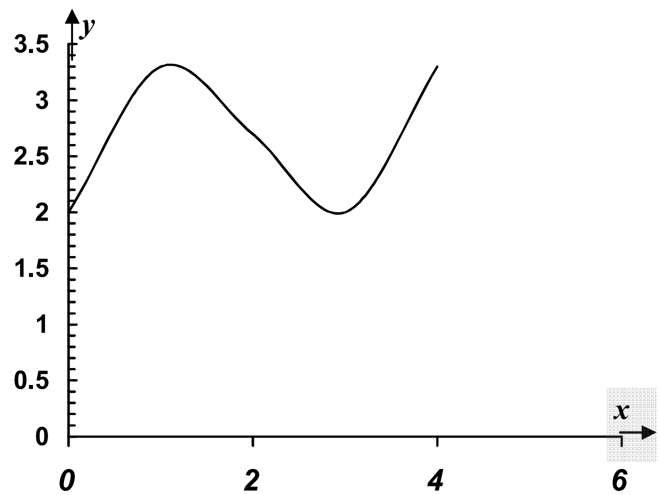



The function has a local maximum at $x = 1$ and a local minimum at $x = 3$ .

To get a more accurate curve , we take :

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $f(x)$ | $2$ | $3.3$ | $2.7$ | $2$ | $3.3$ |

Then the graph of the function is :




***Concave down and concave up :*** The graph of a differentiable function $y = f(x)$ is concave down on an interval where $f'$ decreases , and concave up on an interval where $f'$ increases.

***The second derivative test for concavity :*** The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, concave up on any interval where $y'' > 0$.

***Point of inflection :*** A point on the curve where the concavity changes is called a point of inflection . Thus , a point of inflection on a twice – differentiable curve is a point where $y''$ is positive on one side and negative on other , i.e. $y'' = 0$.

***EX-6*** – **Sketch the curve :** $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$ .

***Sol.*** -

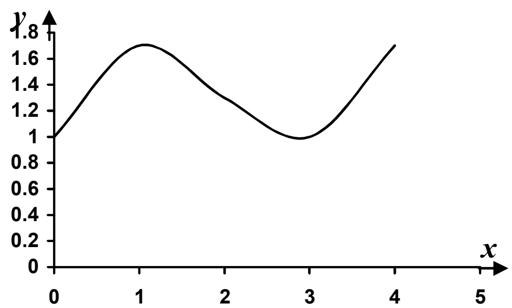
$$y' = \frac{1}{2}x^2 - 2x + \frac{3}{2} = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$$

$$y'' = x - 2 \Rightarrow at\ x = 1 \Rightarrow y'' = 1 - 2 = -1 < 0\ concave\ down\ .$$

$$\Rightarrow at\ x = 3 \Rightarrow y'' = 3 - 2 > 0 \qquad concave\ up\ .$$

$$\Rightarrow at\ y'' = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2\ \ point\ of\ inflection\ .$$

| | | | | | |
|---|---|---|---|---|---|
| *x* | *0* | *1* | *2* | *3* | *4* |
| *y* | *1* | *1.7* | *1.3* | *1* | *1.7* |




***EX-7*** – **What value of** $a$ **makes the function :**

$f(x) = x^2 + \frac{a}{x}$ , **have :**

**i) a local minimum at** $x = 2$ **?**
**ii) a local minimum at** $x = -3$ **?**
**iii) a point of inflection at** $x = 1$ **?**
**iv) show that the function can't have a local maximum for any value of** $a$ **.**

***Sol.*** –

$$f(x) = x^2 + \frac{a}{x} \Rightarrow \frac{df}{dx} = 2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3\ and\ \frac{d^2y}{dx^2} = 2 + \frac{2a}{x^3}$$

i ) at $x = 2 \Rightarrow a = 2 * 8 = 16$ and $\frac{d^2 f}{dx^2} = 2 + \frac{2 * 16}{2^3} = 6 > 0$ Mini.

ii ) at $x = -3 \Rightarrow a = 2(-3)^3 = -54$ and $\frac{d^2 f}{dx^2} = 2 + \frac{2(-54)}{(-3)^3} = 6 > 0$ Mini.

iii ) at $x = 1 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2a}{1} = 0 \Rightarrow a = -1$

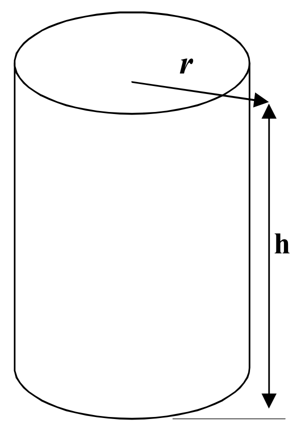
iv ) $a = 2x^3 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2(2x^3)}{x^3} = 6 > 0$

Since $\frac{d^2 f}{dx^2} > 0$ for all value of $x$ in $a = 2x^3$.

Hence the function don't have a local maximum .

**EX-8** – **What are the best dimensions (use the least material) for a tin can which is to be in the form of a right circular cylinder and is to hold *1* gallon (*231* cubic inches ) ?**

**Sol.** – **The volume of the can is :**




$$v = \pi r^2 h = 231 \Rightarrow h = \frac{231}{\pi r^2}$$

**where *r* is radius , *h* is height .**

**The total area of the outer surface ( top, bottom , and side) is :**

$$A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \frac{231}{\pi r^2} \Rightarrow A = 2\pi r^2 + \frac{462}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{462}{r^2} = 0 \Rightarrow r = 3.3252 \text{ inches}$$

$$\frac{d^2 A}{dr^2} = 4\pi + \frac{924}{r^3} = 4\pi + \frac{924}{(3.3252)^3} = 37.714 > 0 \Rightarrow min.$$

$$h = \frac{231}{\pi r^2} = \frac{231}{\frac{22}{7}(3.3252)^2} = 6.6474 \text{ inches}$$

**The dimensions of the can of volume *1* gallon have minimum surface area are :**

***r = 3.3252* in. and *h = 6.6474* in.**

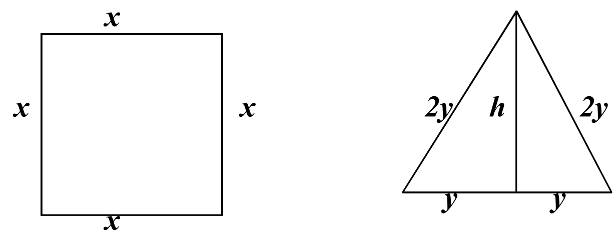
***EX-9*** **– A wire of length $L$ is cut into two pieces , one being bent to form a square and the other to form an equilateral triangle . How should the wire be cut :**

**a) if the sum of the two areas is minimum.**

**b) if the sum of the two areas is maximum.**

***Sol. :*** **Let $x$ is a length of square.**

**$2y$ is the edge of triangle .**




**The perimeter is** $p = 4x + 6y = L \Rightarrow x = \frac{1}{4}(L - 6y)$.

$(2y)^2 = y^2 + h^2 \Rightarrow h = \sqrt{3}y$ *from triangle .*

*The total area is* $A = x^2 + yh = \frac{1}{16}(L - 6y)^2 + y\sqrt{3}y$

$$\Rightarrow A = \frac{1}{16}(L - 6y)^2 + \sqrt{3}y^2$$

$$\frac{dA}{dy} = \frac{-3}{4}(L - 6y) + 2\sqrt{3}y = 0 \Rightarrow y = \frac{3L}{18 + 8\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9}{2} + 2\sqrt{3} > 0 \Rightarrow min.$$

**a) To minimized total areas cut for triangle** $6y = \frac{9L}{9 + 4\sqrt{3}}$

**And for square** $L - \frac{9L}{9 + 4\sqrt{3}} = \frac{4\sqrt{3}L}{9 + 4\sqrt{3}}$ .

**b) To maximized the value of $A$ on endpoints of the interval**

$$0 \le 4x \le L \Rightarrow 0 \le x \le \frac{L}{4}$$

$$at\ x = 0 \Rightarrow y = \frac{L}{6}\ and\ h = \frac{L}{2\sqrt{3}} \Rightarrow A_1 = \frac{L^2}{12\sqrt{3}}$$

$$at\ \ x = \frac{L}{4} \Rightarrow y = 0 \Rightarrow A_2 = \frac{L^2}{16}$$

*Since* $A_2 = \frac{L^2}{16} > A_1 = \frac{L^2}{12\sqrt{3}}$

**Hence the wire should not be cut at all but should be bent into a square .**

# Problems – 4

1. **Find the velocity *v* if a particle's position at time *t* is $s = 180t - 16t^2$ When does the velocity vanish ?** ***(ans.: 5.625)***

2. **If a ball is thrown straight up with a velocity of *32* ft./sec. , its high after *t* sec. is given by the equation $s = 32t - 16t^2$ . At what instant will the ball be at its highest point ? and how high will it rise ?** ***(ans.: 1, 16)***

3. **A stone is thrown vertically upwards at *35* m./sec. . Its height is : $s = 35t - 4.9t^2$ in meter above the point of projection where *t* is time in second later :**
   **a) What is the distance moved, and the average velocity during the $3^{rd}$ sec. (from $t = 2$ to $t = 3$ ) ?**
   **b) Find the average velocity for the intervals $t = 2$ to $t = 2.5$ , $t = 2$ to $t = 2.1$ ; $t = 2$ to $t = 2 + h$ .**
   **c) Deduce the actual velocity at the end of the $2^{nd}$ sec. .**
   ***(ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91, 15.4-4.9h , c) 15.4)***

4. **A stone is thrown vertically upwards at *24.5* m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge *t* sec. later is $4.9t\ (5 - t)$ m. . If its velocity is *v* m./sec. , differentiate to find *v* in terms of *t* :**
   **i) when is the stone at the ledge level ?**
   **ii) find its height and velocity after *1 , 2 , 3* , and *6* sec. .**
   **iii) what meaning is attached to negative value of *s* ? a negative value of *v* ?**
   **iv) when is the stone momentarily at rest ? what is the greatest height reached ?**
   **v) find the total distance moved during the $3^{rd}$ sec. .**
   ***(ans.:v=24.5-9.8t; i)0,5; ii)19.6,29.4,29.4,-29.4;14.7,4.9, -4.9,-34.3; iv)2.5;30.625; v)2.45)***

5. **A stone is thrown vertically downwards with a velocity of *10* m./sec. , and gravity produces on it an acceleration of *9.8* m./sec.$^2$ :**
   **a) what is the velocity after *1 , 2 , 3 , t* sec. ?**
   **b) sketch the velocity –time graph .** ***(ans.: 19.8, 29.6, 39.4,10+9.8t)***

6. **A car accelerates from *5* km./h. to *41* km./h. in *10* sec. . Express this acceleration in : i)km./h. per sec. ii) m./sec.$^2$, iii) km./h.$^2$ .**
   ***(ans.: i)3.6; ii)1; iii) 12960)***

**7.** **A car can accelerate at *4* m./sec.$^2$ . How long will it take to reach *90* km./h. from rest ?** ***(ans.: 6.25)***

**8.** **An express train reducing its velocity to *40* km./h. , has to apply the brakes for *50* sec. . If the retardation produced is *0.5* m./sec.$^2$ , find its initial velocity in km./h. .** ***(ans.: 130)***

**9.** **At the instant from which time is measured a particle is passing through *O* and traveling towards *A* , along the straight line *OA*. It is s m. from *O* after *t* sec. where $s = t\,(t - 2)^2$ :**

**i) when is it again at *O* ?**

**ii) when and where is it momentarily at rest ?**

**iii) what is the particle's greatest displacement from *O* , and how far does it moves , during the first *2* sec. ?**

**iv) what is the average velocity during the *3rd* sec. ?**

**v) at the end of the *1st* sec. where is the particle, which way is it going , and is its speed increasing or decreasing ?**

**vi) repeat (v) for the instant when $t = -1$ .**

***(ans.:i)2;ii)0,32/27;iii)64/27;iv)3;v)OA;inceasing; vi)AO;decreasing)***

**10.** **A particle moves in a straight line so that after t sec. it is *s* m. , from a fixed point *O* on the line , where $s = t^4 + 3t^2$ . Find :**

**i) The acceleration when $t = 1$ , $t = 2$ , and $t = 3$ .**

**ii) The average acceleration between $t = 1$ and $t = 3$ .**

***(ans.: i)18, 54,114; ii)58)***

**11.** **A particle moves along the x-axis in such away that its distance *x* cm. from the origin after *t* sec. is given by the formula $x = 27t - 2t^2$ what are its velocity and acceleration after *6.75* sec. ? How long does it take for the velocity to be reduced from *15* cm./sec. to *9* cm./sec., and how far does the particle travel mean while ?** ***(ans.: 0,-4,1.5 ;18)***

**12.** **A point moves along a straight line *OX* so that its distance *x* cm. from the point *O* at time *t* sec. is given by the formula**
**$x = t^3 - 6t^2 + 9t$ . Find :**

**i) at what times and in what positions the point will have zero velocity .**

**ii) its acceleration at these instants .**

**iii) its velocity when its acceleration is zero .**

**(ans.: i)1,3;4,0; ii)-6,6; iii)-3)**

**13. A particle moves in a straight line so that its distance $x$ cm. from a fixed point $O$ on the line is given by $x = 9t^2 - 2t^3$ where $t$ is the time in seconds measured from $O$ . Find the speed of the particle when $t= 3$ . Also find the distance from $O$ of the particle when $t = 4$ , and show that it is then moving towards $O$ .** *(ans.: 0, 16)*

**14. Find the limits for the following functions by using L'Hopital's rule :**

1) $\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1}$ 2) $\lim_{t\to 0} \frac{\sin t^2}{t}$

3) $\lim_{x\to\frac{\pi}{2}} \frac{2x - \pi}{\cos x}$ 4) $\lim_{t\to 0} \frac{\cos t - 1}{t^2}$

5) $\lim_{x\to\frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x}$ 6) $\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

7) $\lim_{x\to 1} \frac{2x^2 - (3x+1)\sqrt{x} + 2}{x - 1}$ 8) $\lim_{x\to 0} \frac{x(\cos x - 1)}{\sin x - x}$

9) $\lim_{x\to 0} x.\csc^2\sqrt{2x}$ 10) $\lim_{x\to 0} \frac{\sin x^2}{x.\sin x}$

$(ans.: 1)\frac{5}{7};2)0;3)-2;4)-\frac{1}{2};5)\frac{1}{4};6)\sqrt{2};7)-1;8)3;9)\frac{1}{2};10)1)$

**15. Find any local maximum and local minimum values , then sketch each curve by using first derivative :**

1) $f(x) = x^3 - 4x^2 + 4x + 5$ $(ans.: max.(0.7,6.2);min.(2,5))$

2) $f(x) = \frac{x^2 - 1}{x^2 + 1}$ $(ans.: min.(0,-1))$

3) $f(x) = x^5 - 5x - 6$ $(ans.: max.(-1,-2);min.(1,-10))$

4) $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$ $(ans.: min.(0.25,-0.47))$

**16. Find the interval of $x$-values on which the curve is concave up and concave down , then sketch the curve :**

1) $f(x) = \frac{x^3}{3} + x^2 - 3x$ $(ans.: up(-1,\infty);down(-\infty,-1))$

2) $f(x) = x^2 - 5x + 6$ $(ans.: up(-\infty,\infty))$

3) $f(x) = x^3 - 2x^2 + 1$ $(ans.: up(\frac{2}{3},\infty);down(-\infty,\frac{2}{3}))$

4) $f(x) = x^4 - 2x^2$ $(ans.: up(-\infty,-\frac{1}{\sqrt{3}}),(\frac{1}{\sqrt{3}},\infty);down(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}))$

**17. Sketch the following curve by using second derivative :**

1) $y = \dfrac{x}{1+x^2}$ *(ans. : max.(1,0.5); min.(-1,-0.5))*

2) $y = -x(x-7)^2$ *(ans. : max.(7,0); min.(2.3,-50.8))*

3) $y = (x+2)^2(x-3)$ *(ans. : max.(-2,0); min.(1.3,-18.5))*

4) $y = x^2(5-x)$ *(ans. : max.(3.3,18.5); min.(0,0))*

**18. What is the smallest perimeter possible for a rectangle of area *16* in.$^2$ ?** *(ans.: 16)*

**19. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$ .** *(ans.:32)*

**20) A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence . With *800* m. of fence at your disposal . What is the largest area you can enclose ?** *(ans.:80000)*

**21) Show that the rectangle that has maximum area for a given perimeter is a square .**

**22) A wire of length *L* is available for making a circle and a square . How should the wire be divided between the two shapes to maximize the sum of the enclosed areas?**
**(ans.: all bent into a circle)**

**23) A closed container is made from a right circular cylinder of radius *r* and height *h* with a hemispherical dome on top . Find the relationship between *r* and *h* that maximizes the volume for a given surface area *s* .** *(ans. : $r = h = \sqrt{\dfrac{s}{5\pi}}$ )*

**24) An open rectangular box is to be made from a piece of cardboard *8* in. wide and *15* in. long by cutting a square from each corner and bending up the sides Find the dimensions of the box of largest volume .** *(ans.: height=5/3; width=14/3; length=35/3)*