Chapter two
Functions

2-1- Exponential and Logarithm functions :

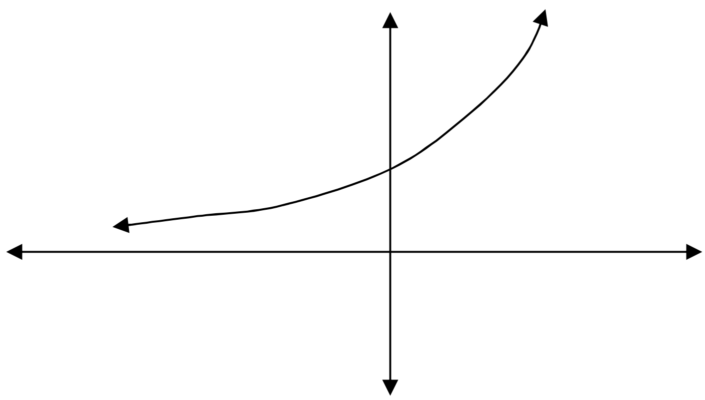
Exponential functions : **If a is a positive number and x is any number , we define the exponential function as :**

$$y = a^x \quad \text{with domain : } -\infty < x < \infty$$
$$\text{Range : } y > 0$$

The properties of the exponential functions are :

1. If $a > 0 \leftrightarrow a^x > 0$.
2. $a^x . a^y = a^{x+y}$.
3. $a^x / a^y = a^{x-y}$.
4. $(a^x)^y = a^{x.y}$.
5. $(a . b)^x = a^x . b^x$.
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$.
7. $a^{-x} = 1 / a^x$ and $a^x = 1 / a^{-x}$.
8. $a^x = a^y \leftrightarrow x = y$.
9. $a^0 = 1$,
   $a^{\infty} = \infty$, $a^{-\infty} = 0$, where $a > 1$.
   $a^{\infty} = 0$, $a^{-\infty} = \infty$, where $a < 1$.

The graph of the exponential function $y = a^x$ is :



Logarithm function : **If a is any positive number other than 1 , then the logarithm of x to the base a denoted by :**

$$y = \log_a x \quad \text{where } x > 0$$

At $a = e = 2.7182828\ldots$, we get the natural logarithm and denoted by :

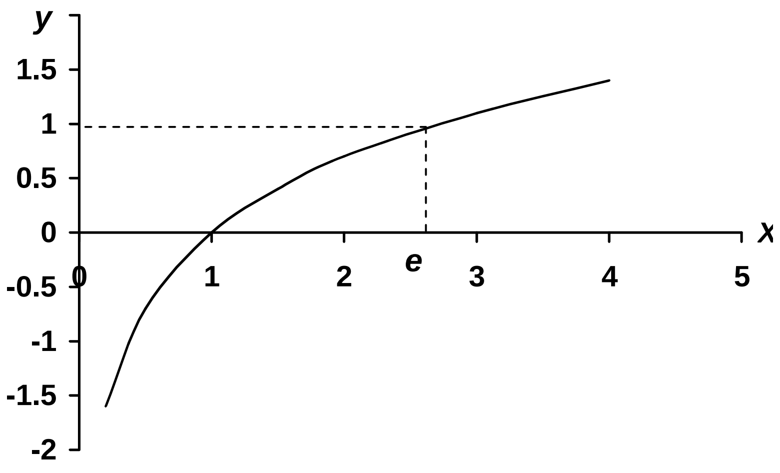
$$y = \ln x$$

Let $x , y > 0$ then the properties of logarithm functions are :

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.

4. $\ln (x.y) = \ln x + \ln y$.
5. $\ln (x / y) = \ln x - \ln y$.
6. $\ln x^n = n. \ln x$.
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x. \ln a}$.
9. $e^{\ln x} = x$.

The graph of the function $y = \ln x$ is :



Application of exponential and logarithm functions **:**

We take Newton's law of cooling :

$$T - T_S = (T_0 - T_S) e^{t k}$$

where T is the temperature of the object at time t .
T_S is the surrounding temperature .
T_0 is the initial temperature of the object .
k is a constant .

***EX-1*-** **The temperature of an ingot of metal is $80\ ^oC$ and the room temperature is $20\ ^oC$. After twenty minutes, it was $70\ ^oC$.**

a) What is the temperature will the metal be after 30 minutes?
b) What is the temperature will the metal be after two hours?
c) When will the metal be $30\ ^oC$?

Sol. **:**

$$T - T_S = (T_0 - T_S)e^{tk} \Rightarrow 50 = 60 e^{20k} \Rightarrow k = \frac{\ln 5 - \ln 6}{20} = -0.0091$$

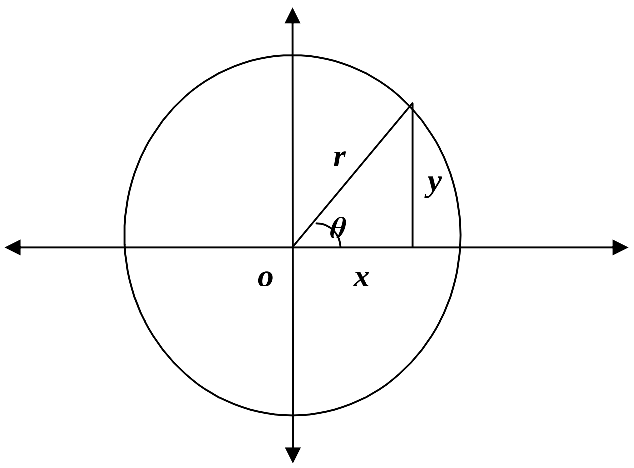
$$a)\quad T - 20 = 60 e^{30(-0.0091)} = 60 * 0.761 = 45.6\ ^oC \Rightarrow T = 65.6\ ^oC$$

$$b)\quad T - T_S = 60 e^{120(-0.0091)} = 60 * 0.335 = 20.1\ ^oC \Rightarrow T = 40.1\ ^oC$$

$$c)\quad 10 = 60 e^{-0.0091\, t} \Rightarrow -0.0091\, t = -\ln 6 \Rightarrow t = 3.3\ hrs.$$

2-2- <u>Trigonometric functions</u> : When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :

$$Sin\theta = \frac{y}{r} = \frac{1}{csc\,\theta} \quad , \quad Cos\theta = \frac{x}{r} = \frac{1}{sec\,\theta} \quad , \quad tan\,\theta = \frac{y}{x} = \frac{1}{Cot\theta} = \frac{Sin\theta}{Cos\theta}$$



The following are some properties of these functions :

1) $Sin^2\theta + Cos^2\theta = 1$

2) $1 + tan^2\,\theta = sec^2\,\theta$ and $1 + Cot^2\theta = csc^2\,\theta$

3) $Sin(\theta \mp \beta) = Sin\theta.Cos\beta \mp Cos\theta.Sin\beta$

4) $Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\theta.Sin\beta$

5) $tan(\theta \mp \beta) = \dfrac{tan\,\theta \mp tan\,\beta}{1 \pm tan\,\theta.tan\,\beta}$

6) $Sin2\theta = 2Sin\theta.Cos\theta$ and $Cos2\theta = Cos^2\theta - Sin^2\theta$

7) $Cos^2\theta = \dfrac{1 + Cos2\theta}{2}$ and $Sin^2\theta = \dfrac{1 - Cos2\theta}{2}$

8) $Sin(\theta \mp \dfrac{\pi}{2}) = \mp Cos\theta$ and $Cos(\theta \mp \dfrac{\pi}{2}) = \pm Sin\theta$

9) $Sin(-\theta) = -Sin\theta$ and $Cos(-\theta) = Cos\theta$ and $tan(-\theta) = -tan\,\theta$

10) $Sin\theta.Sin\beta = \dfrac{1}{2}[Cos(\theta - \beta) - Cos(\theta + \beta)]$

$Cos\theta.Cos\beta = \dfrac{1}{2}[Cos(\theta - \beta) + Cos(\theta + \beta)]$

$Sin\theta.Cos\beta = \dfrac{1}{2}[Sin(\theta - \beta) + Sin(\theta + \beta)]$

$$11)\quad Sin\theta + Sin\beta = 2Sin\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

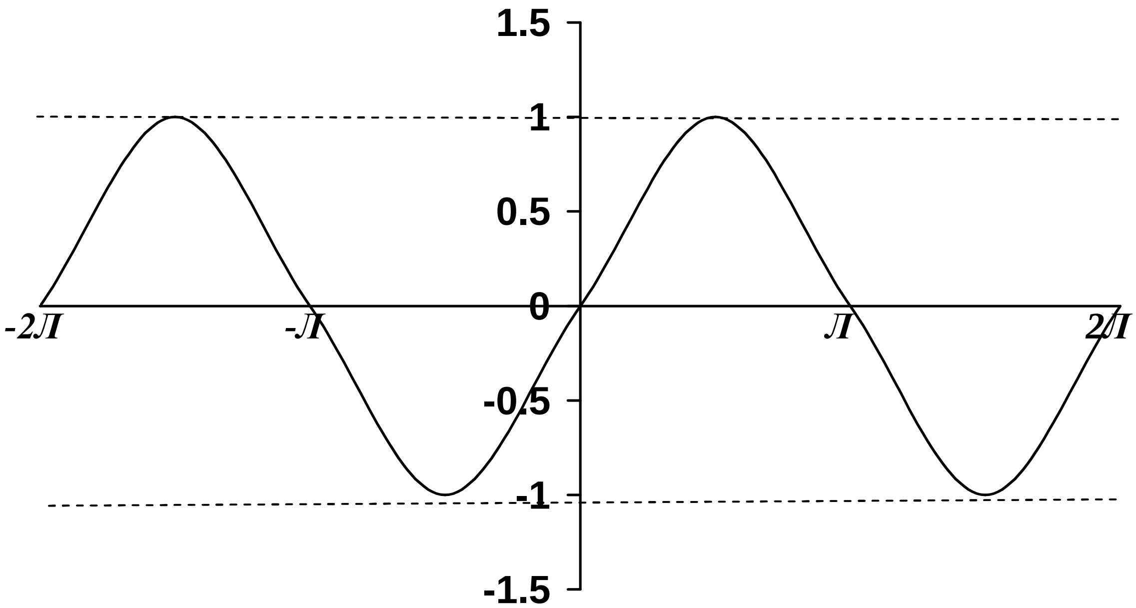
$$Sin\theta - Sin\beta = 2Cos\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

$$12)\quad Cos\theta + Cos\beta = 2Cos\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

$$Cos\theta - Cos\beta = -2Sin\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

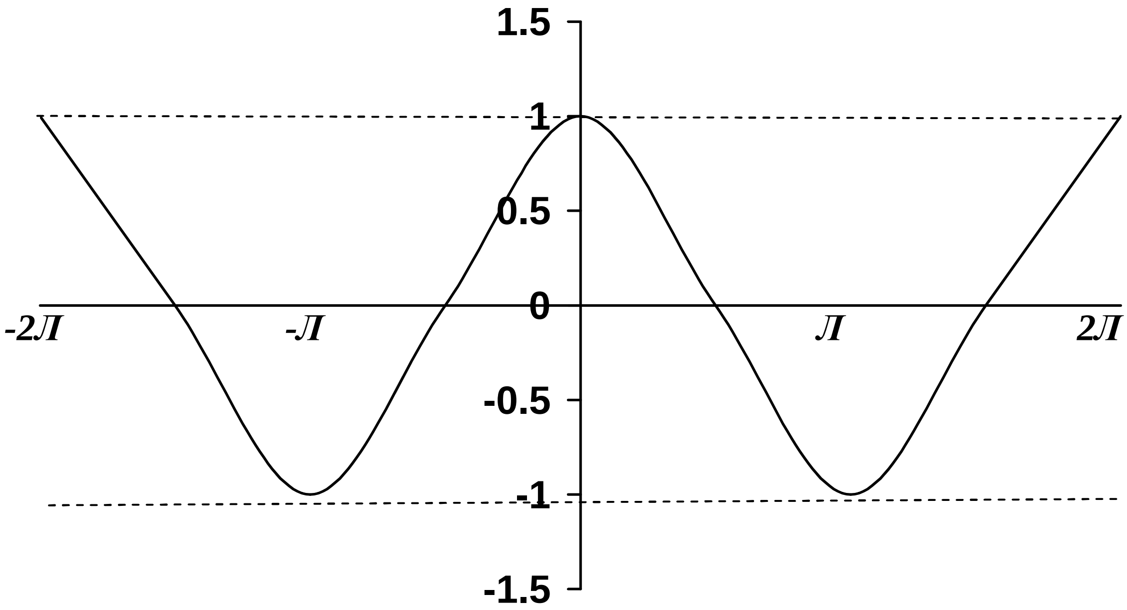
θ	0	Π / 6	Π / 4	Π / 3	Π / 2	Π
$Sin\theta$	0	1/2	1/√2	√3/2	1	0
$Cos\theta$	1	√3/2	1/√2	1/2	0	-1
$tan\theta$	0	1/√2	1	√3	∞	0

Graphs of the trigonometric functions are :



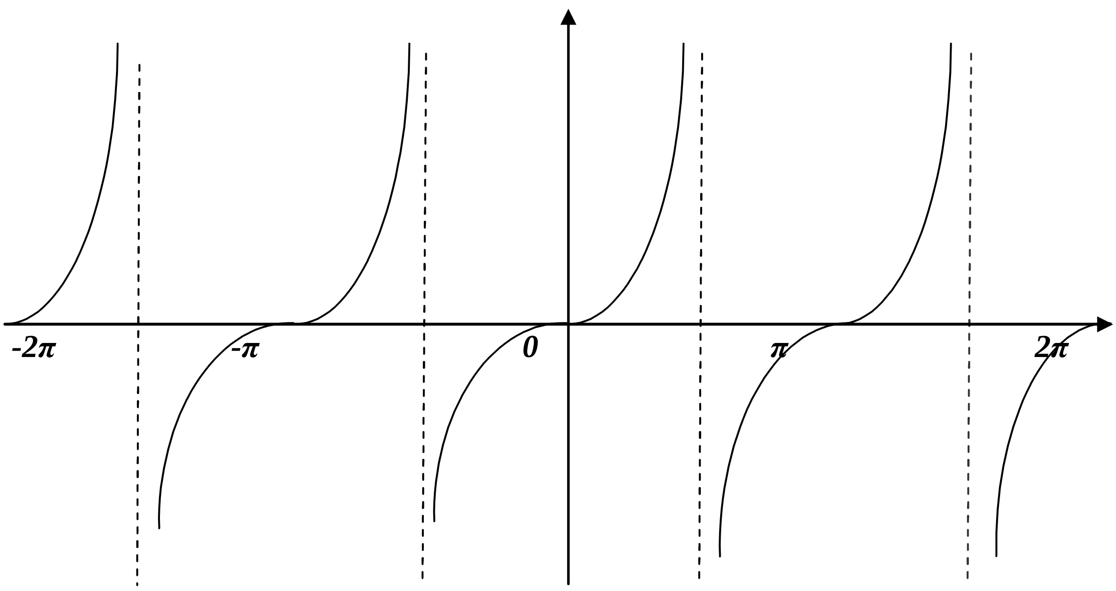
$$y = Sinx \quad D_x : \forall x$$

$$R_y : -1 \le y \le 1$$



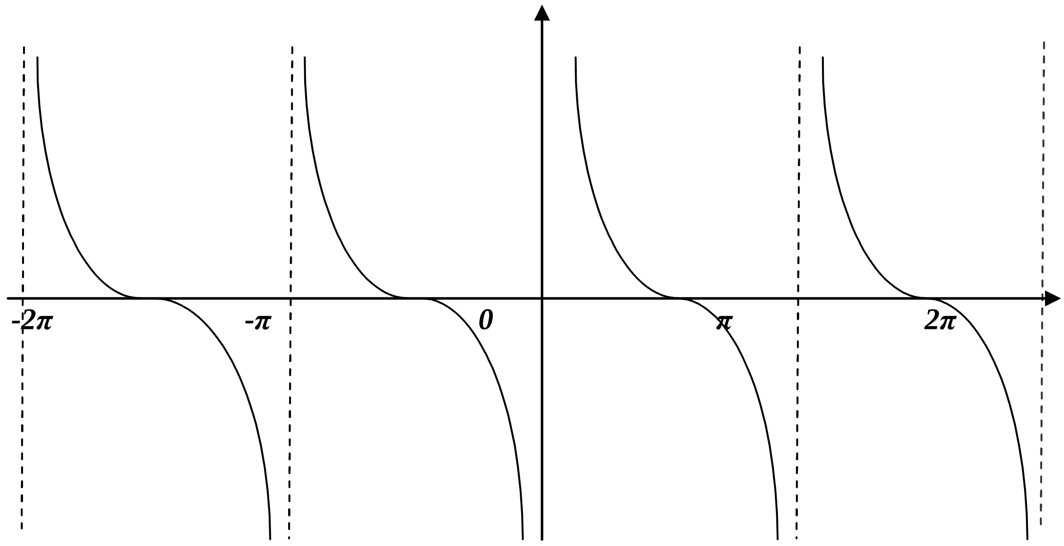
$y = Cosx \quad D_x : \forall x$

$R_y : -1 \leq y \leq 1$

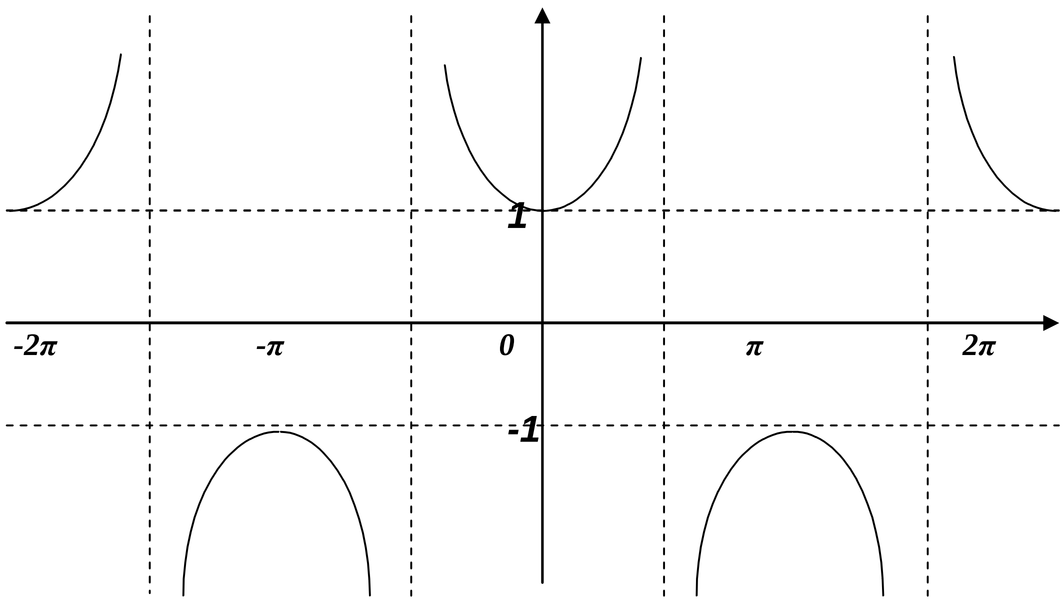


$y = \tan x \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$

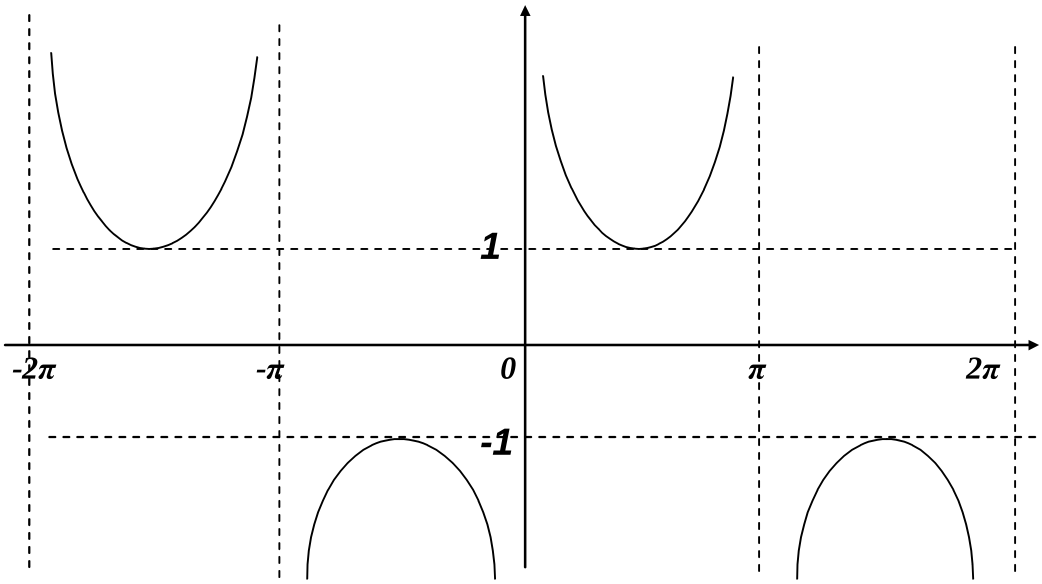
$R_y : \forall y$



$$y = Cotx \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y$$



$$y = Secx \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y \geq 1 \ \ or \ \ y \leq -1$$



$$y = csc\,x \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y \geq 1 \ \ or \ \ y \leq -1$$

$$Where \ \ n = 0, \mp 1, \mp 2, \mp 3, ...$$

<u>*EX-2*</u> - Solve the following equations , for values of *θ* from 0^o to 360^o inclusive .

a) $tan\ \theta = 2\ Sin\ \theta$ **b)** $1 + Cos\ \theta = 2\ Sin^2\ \theta$

<u>*Sol.*</u>-

$$a) \quad tan\theta = 2Sin\theta \Rightarrow \frac{Sin\theta}{Cos\theta} = 2Sin\theta$$
$$\Rightarrow Sin\theta(1 - 2Cos\theta) = 0$$
$$either \ \ Sin\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o$$
$$or \quad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

Therefore the required values of *θ* are $0^o, 60^o, 180^o, 300^o, 360^o$.

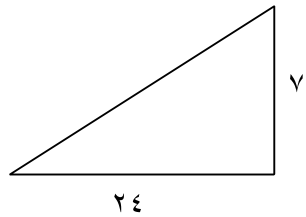
$$b) \quad 1 + Cos\theta = 2.Sin^2\theta \Rightarrow 1 + Cos\theta = 2(1 - Cos^2\theta)$$
$$\Rightarrow (2Cos\theta - 1)(Cos\theta + 1) = 0$$
$$either \qquad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$
$$or \qquad Cos\theta = -1 \Rightarrow \theta = 180^o$$

There the roots of the equation between 0^o and 360^o are $60^o, 180^o$ and 300^o .

EX-3- **If** ***tan θ = 7/24,*** **find without using tables the values of** ***Secθ*** **and** ***Sinθ.***
Sol.-

$$\tan\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$Sec\theta = \frac{r}{x} = \frac{25}{24} \quad and \quad Sin\theta = \frac{y}{r} = \frac{7}{25}$$



EX-4- **Prove the following identities :**

$$a)\quad Csc\theta + \tan\theta.Sec\theta = Csc\theta.Sec^2\theta$$

$$b)\quad Cos^4\theta - Sin^4\theta = Cos^2\theta - Sin^2\theta$$

$$c)\quad \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta}$$

Sol.-

$$a)\quad L.H.S. = Csc\theta + \tan\theta.Sec\theta = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos\theta}.\frac{1}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Sin\theta.Cos^2\theta} = \frac{1}{Sin\theta}.\frac{1}{Cos^2\theta} = Csc\theta.Sec^2\theta = R.H.S.$$

$$b)\quad L.H.S. = Cos^4\theta - Sin^4\theta = (Cos^2\theta - Sin^2\theta).(Cos^2\theta + Sin^2\theta)$$

$$= Cos^2\theta - Sin^2\theta = R.H.S.$$

$$c)\quad L.H.S. = \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\frac{1}{Cos\theta} - \frac{1}{Sin\theta}}{\frac{Sin\theta}{Cos\theta} - \frac{Cos\theta}{Sin\theta}} = \frac{1}{Sin\theta + Cos\theta}$$

$$= \frac{Sin^2\theta + Cos^2\theta}{Sin\theta + Cos\theta}.\frac{\frac{1}{Sin\theta.Cos\theta}}{\frac{1}{Sin\theta.Cos\theta}} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta} = R.H.S.$$

EX-5- **Simplify** $\frac{1}{\sqrt{x^2 - a^2}}$ *when* $x = a.Csc\theta$.

Sol.- $\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{a^2Csc^2\theta - a^2}} = \frac{1}{a\sqrt{Cot^2\theta}} = \frac{1}{a}\tan\theta$.

EX-6- **Eliminate** ***θ*** **from the equations :**

i) x = a Sinθ and y = b tanθ
ii) x = 2 Secθ and y = Cos2θ

Sol.-